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THE ARITHMETIC TEACHER

Volume 8, number 5 MAY 1961

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Let's take a look at division — *Paul A. Hilaire*

Understanding and the ability to solve problems — *Angela Pace*

On the fraction as a numeral — *Francis J. Mueller*

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Suggestions for teaching mathematics to third graders, a summary of references on elementary-school mathematics published during 1960, program high lights of Twenty-First Summer Meeting.

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As we read

How many times in the past month have you asked yourself, "In what ways can we provide a richer and more effective mathematics curriculum for the children in our elementary schools?" We evaluate and re-evaluate our present practices in light of new knowledge, new research, and new ways of developing old ideas. Often it becomes necessary to sacrifice "out-of-date" and no longer useful practices for more purposeful ones. If you are convinced that changes are possible, we invite you to read these provocative articles.

For some time, people have said that many ideas of geometry should begin in the elementary school, not in the secondary school. Although we may be able to identify some of these kinds of experiences in our programs today, Brune's "Geometry in the grades" gives many more practical suggestions, as well as help in developing a point of view regarding the nature of geometric experiences for young children. Many of us may find such experiences lacking in our present programs.

On the other hand, taking "A look at division" with Hilaire and helping children to develop "Understanding and the ability to solve problems" as described by Pace in her research study give some new views on two trouble spots which are very much a part of our arithmetic program. But, is Hilaire suggesting that we place too much emphasis on a technique of computation before we understand the nature of the division process? Apparently there is no easy road in learning to solve problems. Yet, is one plan of systematic instruction to be recommended over another? Are there any kinds of experiences which make greater contributions than do others in developing ability in solving problems? Any good research study raises many more questions, and certainly Pace has.

In present-day thinking about fractions, there are those who view a fraction as a number—as developed in an article in a recent past issue of THE ARITHMETIC TEACHER. Mueller carefully considers the implications of regarding "The fraction as a numeral." He raises some thought-provoking questions regarding operations with numerals and "denominatorless" decimal fractions.

Also, in experimental programs as well as in supplementary materials, we find suggestions for using non-decimal systems of numeration. For those wanting some suggestions in developing materials, read Hamilton's article, "Fad or foundation." Findings in Hamilton's research would indicate that maybe we have been selling our students short on what we could be doing if we would use other symbols and names. We must admit that we commonly have used the decimal symbols and words as much as was conveniently possible. Does this really build foundations? We wonder. Also, would we have similar results for elementary and secondary pupils?

Constantly we are seeking new ways to "add spice" to those necessary phases of learning needed to develop skill and competency in using mathematical ideas. May we suggest you read the articles by Jennings, Keffer, and Powell for ideas that you may wish to try with your children. We believe you will find them helpful in taking the "monotony out of practice." It is surprising what a difference a little imagination can make in motivating our students—and us, too.

Geometry in the grades

IRVIN H. BRUNE *Iowa State Teachers College, Cedar Falls, Iowa*
Dr. Brune is professor of mathematics at Iowa State Teachers College.

The proper study of mankind is geometry. Tiny Timothy chose a nickel rather than a dime, but it was worldly wisdom, rather than geometric sense, that he lacked. Timothy's grandfather, however, who was shrewd in money matters, believed that three-foot cubes rather than three-foot spheres should adorn the new courthouse. He surmised that the cubes, covered only on five faces, would require less gold leaf to gild. His geometric guess was almost as naive as Timothy's money muddle.

Myra in Grade 5 wanted to grow flowers. Her parents gave her fifty feet of aluminum edging and told her that she could have all the ground that the edging would enclose. What shape of flower bed would give her the most space?

At a sale Mr. Handyman bought a piece of linoleum that was nine feet wide and sixteen feet long. He knew that with only one cut he could fit it on a floor twelve feet by twelve feet. This, as well as the other cases cited, requires geometry.

Or consider the circles in Figure 1. Obviously the black circle on the right is larger than the black circle on the left. But is it?

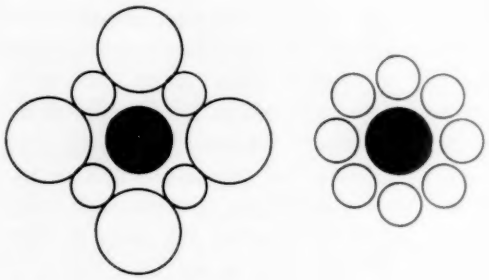


Figure 1

Similarly for Figure 2. Which has the greater area, the outer black annulus (or ring) or the inner black circle?

Figure 2

Another instance is the well-known Figure 3. Which segment, the horizontal or the vertical, is the longer?

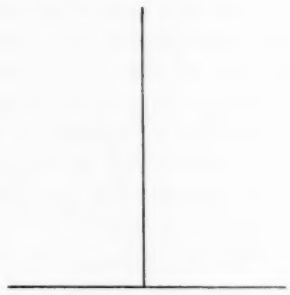


Figure 3

Perhaps you think such examples are trivial. At least they are homely, ordinary. But there are more important applications too; you see them daily. When mathematicians use higher geometry to solve complicated problems in the production and distribution of goods, no one concerned thinks the matter trifling. Consider, for example, the problem of selecting the most economical combinations of some twenty ingredients that fluctuate in price daily. This problem arises in sausage-making. Which quantities of which meats will satisfy fixed standards of high quality

and at the same time cost the least according to today's prices? Here the mathematician employs geometry in an untrivial way.

Ageless geometry

To enumerate and to describe man's uses of geometry would take a trip in time from prehistory to the present moment. The subject began in earth-measuring, it grew in planet-observing, it led the way in pure mathematics, and it pioneered in modern mathematics.

Man has always needed geometric principles, however dimly he may at first have perceived them. Similarly, children's lives cannot be devoid of geometry, however unaware they may be of its formal aspects. For, irrespective of its many applications and regardless of its value as a system of reasoning (and both of these phases merit attention), geometry embodies numerous ideas interesting in themselves.

Geometry for all

We suggest, therefore, that geometry deserves a lifetime of interest. To study it in only the tenth grade hardly suffices. At that level pupils presumably study one or more kinds of geometry as deduction. There and in subsequent courses they also learn about applications. But the computing with geometric formulas that frequently represents the only planned experience that pupils have in geometry prior to Grade 10 seldom prepares them for Grade 10.

Grade-school geometry

Informal geometry in the elementary grades can, therefore, counteract a serious deficiency. In these grades geometry is the study of form. Shapes, sizes, patterns, designs—these are the stuff from which children form concepts. From studying forms children discover numerous geometric relations; from making constructions pupils learn about geometric facts; from measuring figures learners acquire a background of geometric information. The

Figure 4

work teems with both classical concepts and contemporary concepts.

Fun *and* future

We believe, therefore, that children of all ages should get ample opportunities to find out things about geometry. The goal is satisfaction, here and now, with things mathematical, and geometry abounds in such ideas. An accompanying benefit will doubtless be a preparation for a more formalized geometry of proofs. Just as the ancient Egyptians' surveying and the early Chaldeans' star-studying opened a path for the deductions of the Greeks, so the block-arranging of the curious kindergartners and the design-drawing of the enthusiastic upper graders provide understandings for the problems of the older pupils. The pleasures of the moment outweigh the preparations for the future.

Therein lies the heart of the matter. Teachers cherish in their pupils such traits as alertness, preparedness, and willingness. And possibly the greatest of these is willingness. Seldom, though, do these traits develop overnight; rather, they seem to stem from many things that pupils do. Through the situations that teachers encourage them to explore, pupils discover relations, achieve insight, and gain satisfactions for the moment as well as for later studies. Mathematics, you know, is a cumulative subject. For example, clusters of dots, such as those in Figure 4, provide numerous helpful experiences. For infants the dots in Figure 4 are many, whereas those in Figure 5 are few.

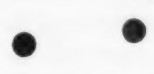


Figure 5

Figure 6a

Figure 6b

Figure 6c

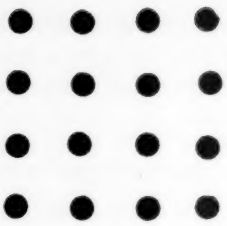


Figure 6d

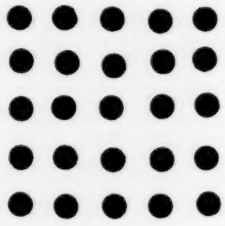


Figure 6e

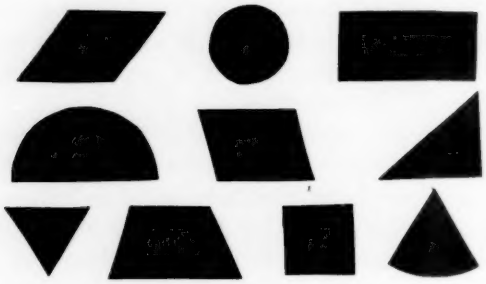


Figure 7

Figure 8a

Figure 8b

Figure 8c

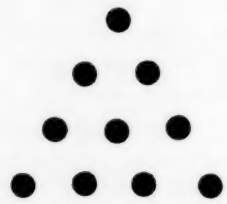


Figure 8d

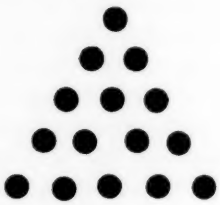


Figure 8e

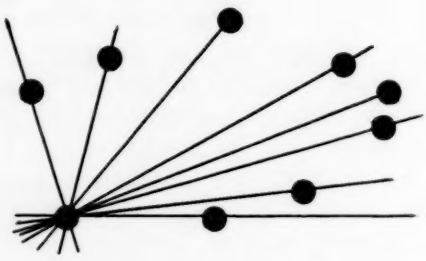


Figure 9

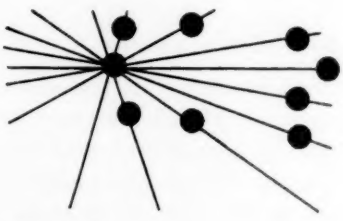


Figure 10

Later the dots represent nine. Then they help with the ancient idea that some numbers are squares: that nine corresponds to three threes. Furthermore, in Figures 6*a–e*, the square arrays, one, four, nine, sixteen, twenty-five, etc., when ordered and compared via differences, encourage pupils to think about the odd numbers.

Square numbers:

—— 1 4 9 16 25 36 49 ...

Differences between successive square numbers:

—— 3 5 7 9 11 13 ...

Does zero belong in the blank before 1 in the top row?

For upper graders, finding a continuous path containing four line segments and connecting all the nine points will probably be a fascinating challenge. It might also be preparation for later work in simple topology, where, among other things, the study of paths—closed paths and not-closed paths—receives attention.

Geometric readiness

Examples such as the foregoing abound. Suppose that we look briefly at the pupils' readiness before we consider further instances.

We have hinted that readiness in geometry implies at least two other "nesses":

1. Preparedness, or adequate mathematical maturity to go on.
2. Willingness, or enough emotional security to go on.

Geometric preparedness begins at an early age. Tots in kindergarten enjoy plopping the cutout figures into their proper places (Fig. 7). Children quickly discriminate between right triangles and equilateral triangles, between squares and oblongs, and between trapezoids and parallelograms. Already these youngsters are shape conscious.

Success in this sort of activity leads young children on. Their handling of squares, cubes, disks, triangles, spheres, and so on, prepares them for further work with forms.

All too frequently, however, such activities terminate abruptly. This occurs because courses of study encourage the pupils to put away "childish things" and settle down to the stern business of memorizing facts and practicing operations with numbers. Since *perfection* in these worthy matters eludes most learners, the study of facts and operations flourishes while the study of forms languishes. Of course, lessons in the upper grades deal with areas and volumes, but computing with numbers and distinguishing between area and perimeter and between volume and surface have been known to monopolize the act.

Fortunately, the trend today points to geometry for the sake of geometry, rather than to geometry as further practice in calculating. In the elementary grades informal, or intuitive, studies get the emphasis. Drawing, counting, and measuring lead pupils to observing, inferring, and generalizing. Consciousness of forms continues to grow, and readiness for proofs in geometry also continues to grow.

Let us return momentarily to the tots. By handling wooden, paper, or plastic representations of geometric figures, children appreciate numerous ideas; among these are notions of square corners, straight edges, round edges like faces, roundness of disks, and roundness of balls. These children gain a degree of understanding to go on; they grow in geometric readiness.

But children gain in willingness too. The shapes, the fitting of objects into patterns, the matching, the comparing, and the counting all make children receptive to further activities. One quite ordinary first grader happened onto triangular-number arrangements, as in Figures 8*a–e*. Pupils do things, learn, and crave to learn some more.

So, as they progress in mathematical maturity (preparedness), pupils tend to seek new mathematical worlds to conquer (willingness). Thus, willingness and also preparedness stem from activities—things done successfully. Junior is likely to wonder what the next step will lead to. If, for example, thirty-six lines can be drawn through nine points such that no three points lie on a line (Fig. 9), then how many lines can be drawn if exactly three of the nine are on one line? (Fig. 10.) In the

figures the joins of one point with each of the others constitute a hint; this is one way to begin the problems.

In sum, teachers seek to challenge pupils, not to frustrate. Sometimes a team approach (or working as a class on a perplexing problem) will prevent defeat. Whether pupils suffer more from frustration than from boredom, however, is moot.

Some simple arrangements

Besides patterns of points previously mentioned, we might look at a few other configurations. In Figure 11 a side of the square $ABCD$ measures 3, AE measures 2, and angle DEF measures 90°. In square $KLMN$, a side measures 2, and angle LKP has the same measure as angle BEF. The sections thus cut form two separate squares or one combined square—a kind of readiness for the Theorem of Pythagoras.

In Figure 12 $ABCD$ is a square with a side that measures 6, and $BEFG$ is a square with side 3. $AL = BH = CJ = DK = 1\frac{1}{2}$, and BN measures 3. M is the intersection of HK and JL. Square $MNOP$ has the same measure in square units as squares $ABCD$ and $BEFG$ combined.

Now suppose that we begin with other segments, half-squares and half-oblongs, for example. These, plus a few rectangles, semicircles, and quadrants, form a variety of designs. Figures 13–17 illustrate some of them.

Abundant triangles

If the pupils start with three equal segments and the question, "What can we do with the segments?", they can soon come up with an equilateral triangle (Fig. 18). By joining the mid-points of the sides of the triangle, they can produce four equilateral triangles (Fig. 19). By repeating the process successively for unshaded triangles, the pupils obtain Figures 20 and 21.

Or, the pupils may choose not to shade any of the component triangles and proceed to successive quartering of all the new triangles. One triangle yields four triangles, four yield sixteen, and sixteen yield sixty-four. Some pupils will move ahead and forge a fifth stage or even further proliferations of triangles. Theoretically, the fast workers need not grow weary of waiting for their slower classmates to finish a step. Endless steps await those who wish them, and the steps get harder.

Some pathological curves

If the pupils begin again with three equal segments, they can form another equilateral triangle (Fig. 22). By trisecting the sides, erecting equilateral triangles on the middle sections, and erasing the intersections of these four triangles, the pupils get Figure 23. Then in Figure 24 further trisections and outward points appear. Some pupils may wish to carry this procedure still further. Although the area of this snowflake-like curve clearly cannot exceed the surface of the page, its perimeter becomes infinitely large.

Similarly for other pathologic curves (Figs. 25–27), the pupils proceed from an equilateral triangle again. Here, however, the open mid-sections are spanned by equal segments that intersect inside instead of outside the original triangle. This gives an inverted snowflake pattern. Here too, the perimeter can be made infinitely large, even though the area will not exceed that of the drawing paper.

Figures 28, 29, and 30 show what results when pupils begin with a circle, divide it into six equal parts, and invert alternate arcs. This procedure, repeated, leads to another figure, the aesthetics of which may be doubtful. It is known as an inside-outside curve. It troubles almost everyone who seeks to determine its curvature.

Tiling patterns

As pupils soon learn when they begin to work with measures of angles, one full turn measures 360°. A further interesting investigation results when pupils face the question, "What flat figures will fit around

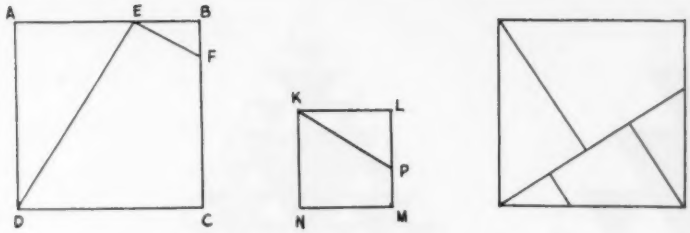


Figure 11

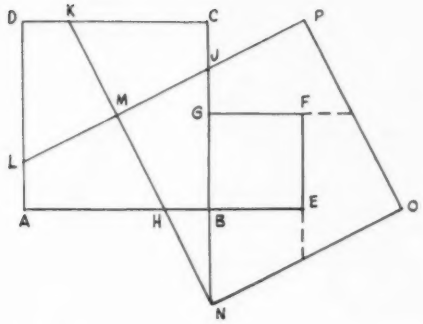


Figure 12

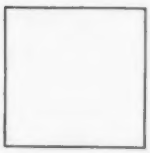


Figure 13

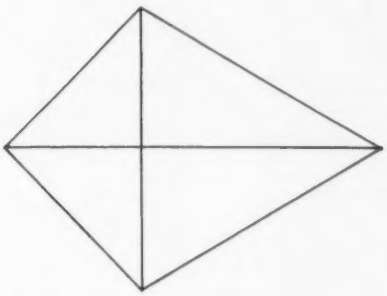


Figure 15

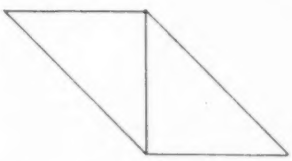


Figure 14

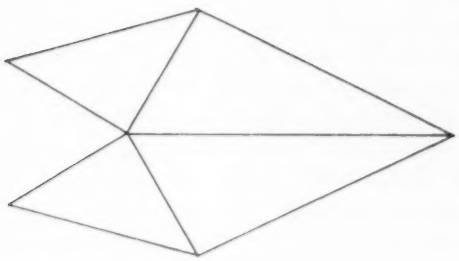


Figure 16

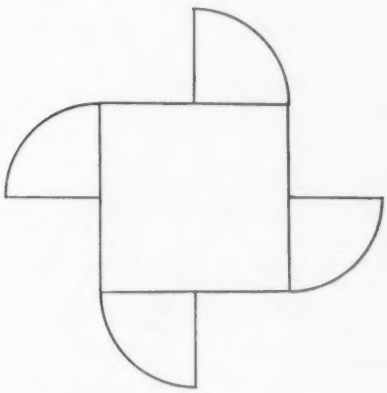


Figure 17

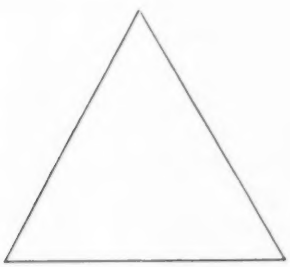


Figure 18

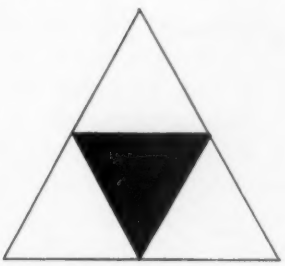
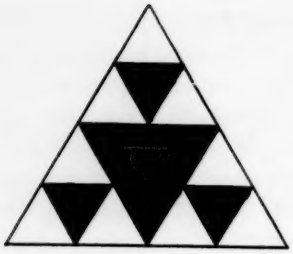


Figure 19



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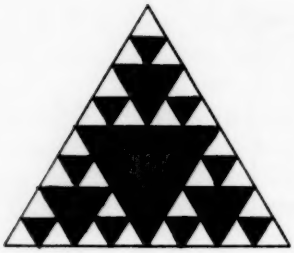

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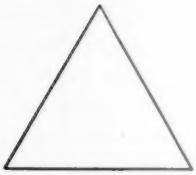

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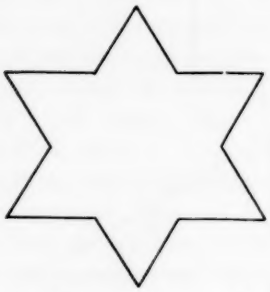

Figure 23

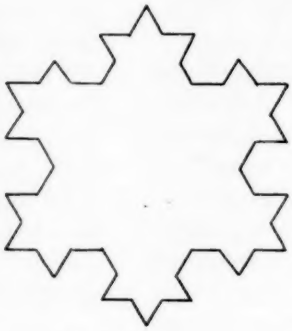

Figure 24

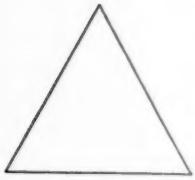

Figure 25

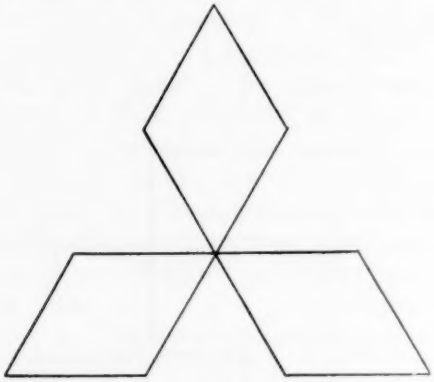

Figure 26

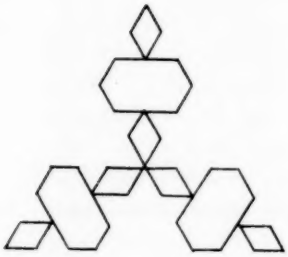

Figure 27

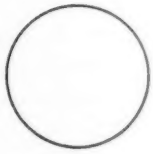

Figure 28

Figure 29

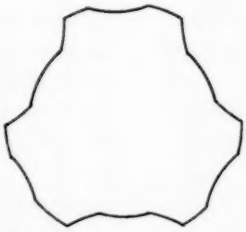

Figure 30

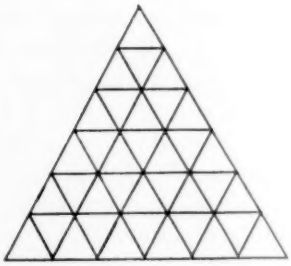

Figure 31

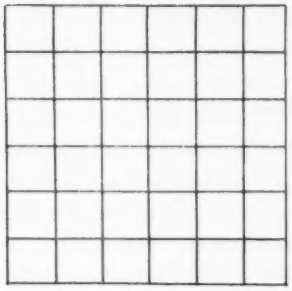

Figure 32

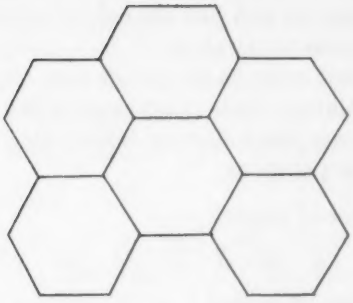

Figure 33

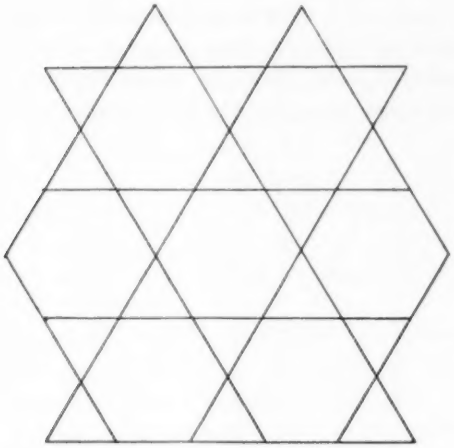

Figure 34

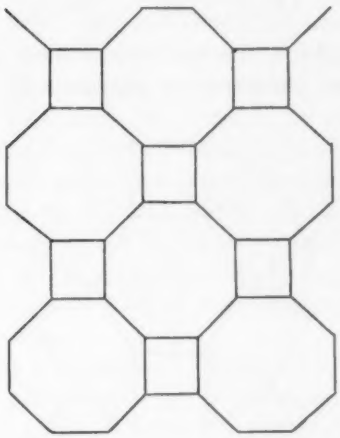

Figure 35

a point and fill in the flat surface?" Regular polygons seem to be needed, although all rectangles will suffice, but not all regular polygons will meet the requirement.

Considerable acquaintance with these polygons can result from such experimenting. How can we draw them? Later the pupils will study the straightedge-compass constructions for regular polygons, and still later they will study criteria of constructibility.

But strictly informal experimenting will reveal some combinations of polygons that, so to speak, cover the floor. Indeed, among sophisticates the whole subject of floor coverings, tilings, and mosaics bears the impressive name of "tessellation."

Six equilateral triangles (Fig. 31), four rectangles (Fig. 32), and three hexagons (Fig. 33) exhaust the possibilities of how many flat figures will fit around a point and fill in a flat surface. If, however, the

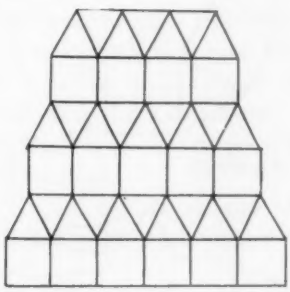

Figure 36

pupils do not limit the problem to polygons of one single sort, then the following serve: two hexagons and two triangles (Fig. 34); two octagons and one square (Fig. 35); three triangles and two squares (Fig. 36); one hexagon, two squares, and one triangle (Fig. 37). Still other possibilities, not illustrated here, exist: one hexagon and four triangles; one dodecagon, one hexagon, and one square; two

dodecagons and one triangle. Your pupils may want to try them.

It will occur to the pupils that these are possibilities when they experiment and construct the following table, referring to regular polygons:

Number of sides:

3 4 5 6 7 8 9 10 11 12

Measure of angles:

60	90	108	120	$128\frac{4}{7}$
135	140	144	$147\frac{3}{11}$	150

From the increasing sizes of the angles, it appears that regular polygons having a still greater number of sides are not likely candidates.

Centroids

Locating a centroid, or a center of mass of a body, can become a thorny problem. Quite young children, on the other hand, can cut geometric forms from cardboard and readily locate lines and points of balance.

Regular figures balance rather easily on a knife-edge. The intersection of two such lines of balance determines a center of balance, or a center of mass. Equilateral triangles, squares, regular hexagons, reg-

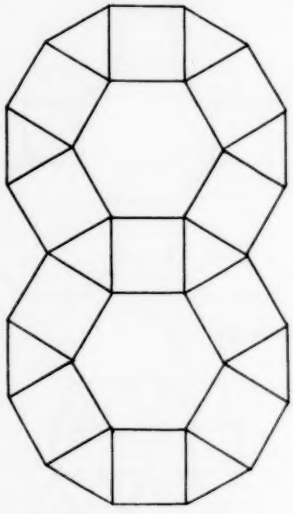

Figure 37

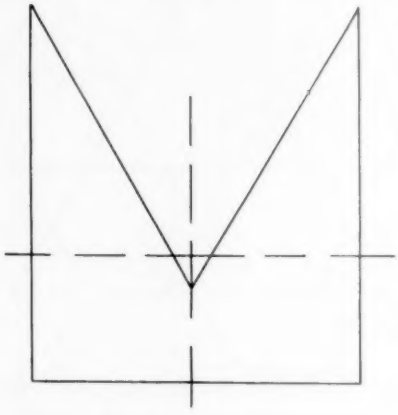

Figure 38

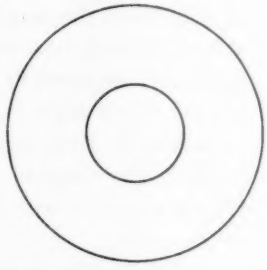

Figure 40

Figure 39

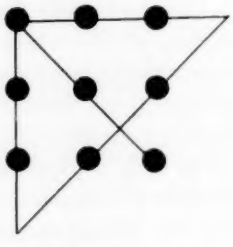

Figure 41

ular octagons, and circles illustrate these ideas.

When the figures depart from regularity, the knife-edge procedure may become more difficult, as in the case of a convex polygon. There the centroid lies outside the figure (Fig. 38). This might become the germ of the idea of a centroid for a *system* of bodies, which may interest the future physicists and astronomers in your classes.

Simple and nonsimple

Finding the centroid of an involved, yet technically simple, curve could provide a difficult problem. A cutout again provides an intuitive approach (Fig. 39). Incidentally, when is a curve simple? If the left circle were removed from Figure 39, the curve would remain simple. But if the right circle were removed, the figure would become nonsimple.

Figure 40 is another puzzler. It has two outsides, one of which might seem at first to be inside. When is an outside inside?

In case you wondered, here is one solution to the path problem, referred to earlier, for nine points arranged as three threes (Fig. 41).

Yin-yang

Rooted in antiquity, especially in venerable Chinese philosophy, is the symbol represented in Figure 42. From *yang*, literally the south or sunny side of a hill, the unshaded portion of the design represents the bright, good, positive, male principle in Chinese dualism. *Yin*, on the contrary, symbolizes the dark, evil, negative, and female. The shaded part stands for *yin*.

Figure 42

Regardless of how pupils and their teachers look on such mystical matters, the emblem appears to be easy to construct and to describe geometrically.

Summary

Suppose, now, that we summarize. From the kindergarten on the concepts, rules, and operations of arithmetic and algebra dominate pupils' experiences in mathematics. Few deny it, and if the teaching has been good, still fewer bemoan it.

New occasions, however, teach new duties. Mathematics grows apace; mathematics education accelerates its search. With reason we urge teachers to learn new mathematics and teach new courses. We see merit in helping young children to gain acquaintance with good mathematics early. We note with pleasure the improvements in textbooks of elementary-school mathematics.

In our zeal, however, we run the risk of letting abstractions get out of hand. This we disapprove. The motto "be abstract" should be left to the habitants of Greenwich Village. The race, including those of us who urge mathematics reforms, learned mathematics through practical needs and real problems. The abstractions, the generalizations, and the deductions followed the investigations, the approximations, and the corrections.

Surely, therefore, we should not deny young children the opportunity to explore and learn.

A proper study for all children is geometry—the geometry of form. Here pupils perceive, compare, measure, and generalize. Here they sharpen intuition without plunging too far into abstractions. Above all, children see values in what they do. If we can encourage pupils to discover for themselves some principles in the science of space, then they will bring into their geometry classes a usable store of information about the Euclidean plane. They might also have a good start on three-space.

Let's take a look at division

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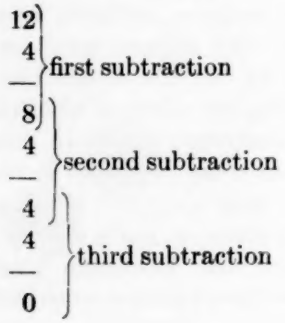
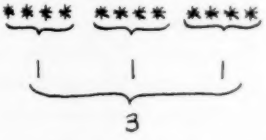
Children find division difficult to master. Teachers find it difficult to teach. "The teacher who feels insecure, who dreads and dislikes the subject, for whom arithmetic is largely rote manipulation, devoid of understanding, cannot avoid transmitting her feelings to the children. . . . Many more teachers who fail as arithmetic teachers do so because of their own inadequacy with the subject than from a lack of knowledge of and skill in methods of teaching or from inability to work successfully with children."[1]

If the above statements are true, then perhaps an analysis of the rationale of division, aiming for an understanding of the "what" and the "why," may benefit teachers of arithmetic.

Division may be considered as an "undoing process." Like subtraction, it is associated with the partition of sets of things. In fact, division is often considered to be "repeated subtraction" when all subtrahends are equal. For example:



Since the operations in the previous example can be performed, we say $12 \div 4 = 3$. One is actually determining how many subsets of four elements each may be found in a set containing twelve elements.

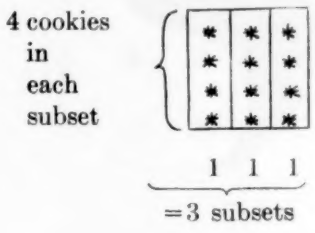


There is also an important relationship between division and multiplication. Just as subtraction is the inverse of addition, division is the inverse process of multiplication. $12 \div 4 = 3$ because $3 \times 4 = 12$; or,

$$\frac{12}{4} = 3 \quad \text{because} \quad 3 \times 4 = 12.$$

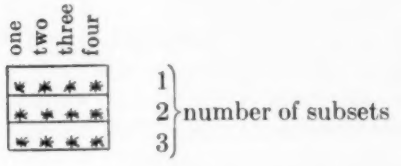
Division is often defined as "the operation of finding either of two factors when their product and the other factor are given."[2]

Division is used to solve two different kinds of problems. The first may be illustrated by the problem: "How many sets of four cookies each are contained in a set of twelve cookies?"



This is commonly known as "measurement division." The analysis involves action in which a set of things is put into subsets. In this example the separating is based upon each subset containing the same number of elements. Note that if one thinks of the inverse operation, $?\times 4=12$, the multiplier is being sought. The problem seeks to find "how many subsets" when the number in each subset is a fixed quantity.

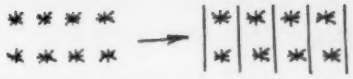
The second kind of problem may be illustrated by the problem: "How many cookies are in each subset if twelve cookies are divided into three equal subsets?"



Here a collection is being separated into a number of equal subsets. The situation is partitive. It requires partitioning or separating, and is commonly referred to as "partitive division." Again, if one looks at the inverse operation, $3\times ?=12$, it is evident that the multiplicand is being sought. It seeks to find "How many in each subset" when the number of subsets is fixed.

The algorithm for each of these problems is the same. Nevertheless, one must be aware that in life there are two different types of problems which may be solved by the process of division. Children should be made aware of these differences before the algorithms are taught. This may be accomplished by using experiences within the understanding of the child. For example:

1. Cut a length of ribbon into given lengths.
2. Divide a number of apples among a given number of students.
3. Determine how many subgroups, each containing 2, are in the group of 8 shown at the top of the next column.



4. Determine the truckloads, each capable of holding 4 tons, required to move 13 tons of dirt. Note that in this case a different problem is faced. It requires 4; 3 will not do.

There is another aspect of mathematics which should be understood. Are natural numbers "closed" or "not closed" with respect to division? Closure is defined as that property of a set in which any two of the members of the set under a given operation yield a member of the set. If two counting numbers are added, the sum is always another counting number. Counting numbers are, therefore, considered "closed" with respect to addition. On the other hand, if one tries to divide any counting number by another, the result does not always fall within the classification of counting numbers. Therefore, counting numbers are considered as "not closed" with respect to division. If our given set of reference, the counting numbers, is expanded to include fractions of the form a/b, $b\neq 0$; then when division of one member of the set by another is performed, the result will always be within the reference set, and this new set with reference to division is said to be "closed."

You say that this is complicated. Certainly, it is. That is why one must understand and know what is involved when division is performed. Is it not wonderful that children *do* learn what is involved in division?

To make matters worse, consider these:

1. $2+3$ is read "two plus three"; $5-1$ is "five minus one"; 6×4, "six times four"; and $8\div 2$, "eight divided by two." $2+3$ can be $3+2$, but $5-1$ cannot be $1-5$ nor can $8\div 2$ be $2\div 8$. But 6×4 can be 4×6. In other words, addition and multiplication are commutative, but division and subtraction are not commutative.

2. A law of compensation works in division. Consider this:

$36 \div 12 = 3$. (Original problem.)
$18 \div 6 = 3$. (Dividend and divisor
$12 \div 4 = 3$. may be divided by the same number and the quotient remains the same.)

$72 \div 24 = 3$. (Dividend and divisor
$108 \div 36 = 3$. were both multiplied by the same number and quotient did not change.)

3. $36 \div 9 = 36 \times \frac{1}{9} = 4$. (Dividing by a number is the same as multiplying by its reciprocal.)

4. Division is distributive with respect to addition, $(16+12+8) \div 4 = 36 \div 4 = 9$, but the same result may be obtained by dividing each addend by four and then finding the sum.

$$\frac{16}{4}+\frac{12}{4}+\frac{8}{4}=4+3+2=9.$$

5. One is the identity element in division.

$$8 \div 1 = 8, \qquad 16 \div 1 = 16;$$

in fact,

$$N \div 1 = N \quad \text{and also} \quad \frac{N}{N}=1, \quad N \neq 0.$$

Any number may be divided by one, and its value does not change; or any number divided by itself equals one.

6. Zero divided by any nonzero number is zero.

$$\frac{0}{6}=0 \quad \text{because} \quad 0(6)=0.$$

$$\frac{0}{12}=0 \quad \text{because} \quad 0(12)=0.$$

$$\frac{0}{N}=0 \quad \text{because} \quad 0(N)=0.$$

7. Division of a number by zero is undefined.

$$\frac{N}{0}=Q.$$

No number can be given as an answer because no matter which number is used, it would have to be multiplied by zero. The product, therefore, could never equal a number other than zero. $Q(0)=0$. $Q(0) \neq N$.

8. Zero divided by zero is indeterminate.

$$\frac{0}{0}=?$$

The answer could be 6, 15, 1286, or any number (N). This result, N, when multiplied by zero always produces zero, $N(0)=0$. Who knows which number to choose for an answer? Therefore, zero divided by zero is said to be indeterminate.

These last three facts may be postponed until a child is more mature, since it is not likely that such meaning will come within the experiences of young children. It seems foolish to ask, "How many dozen oranges in an empty sack?"

$$\frac{0}{12}=?$$

However, in maturity, a child may be given experiences which will aid him in rationalizing these concepts. These experiences will, no doubt, involve the concept of division as the inverse of multiplication, since

$$\frac{0}{N}=0 \quad \text{because} \quad N(0)=0.$$

While it is true that we want children to master the basic division facts, it is neither necessary nor advisable to deal with them as separate experiences. When children are working with concrete objects, such as blocks, several concepts can be experienced. For example:

shows

(1) $3+3+3=9$.
(2) $3 \times 3 = 9$, or $3(3)=9$, or $3 \cdot 3 = 9$.

(3) $9 \div 3 = 3$, or

$$\frac{9}{3} = 3, \quad \text{or} \quad 3\overline{)9} \quad \text{or} \quad \underline{3}/\overline{9},$$

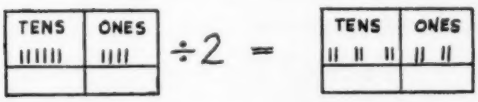
or $9/3 = 3$.

(4) $\square + (3+3) = \square + 6 = 3$.

(5) $9 - 3 = (3+3) = 6$.

Initial instruction with the algorithm of division should stress two mathematical principles, namely, the distributive property and "repeated subtraction." Consider the example:

I $\quad \underline{2}/\overline{64}.$



One reasons that in six tens there are three groups of two tens each and that in four ones there are two groups of two ones.

II

Tens	Ones
3	2
$\underline{2}/\overline{6}$	4

reasoning that six tens divided by two gives 3 groups of tens, each containing two tens. Four ones divided by two gives 2 groups of ones, each containing two ones.

III

$$\underline{2}/\overline{64} = \underline{2}/\overline{60+4} = \underline{2}/\overline{60} + \underline{2}/\overline{4}$$

$$= 30 + 2 = 32.$$

IV

$$\begin{array}{r} 2 \\ 30 = 32 \\ \underline{2}/\overline{64} \\ \underline{60} \\ 4 \\ \underline{4} \end{array}$$

reasoning that thirty twos, or sixty, may be subtracted first, then two twos or four.

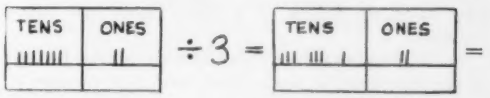
One should realize that the concept, 64 divided by 2, can be written several ways:

$$\underline{2}/\overline{64} = \frac{64}{2} = 2\overline{)64} = 64 \div 2 = 64/2.$$

One should also establish the fact that the dividend may be separated in a variety of ways ($\underline{2}/\overline{64} = \underline{2}/\overline{60} + \underline{2}/\overline{4} = \underline{2}/\overline{32} + \underline{2}/\overline{32}$

$= \underline{2}/\overline{50} + \underline{2}/\overline{14} = \underline{2}/\overline{36} + \underline{2}/\overline{28}$, etc.).

I **Consider $72 \div 3$.**



two groups of three tens, but one ten remains. What can be done?

Answer: Change the one ten to ten ones, giving with the two ones, twelve ones to be divided by 3.

II

$$\underline{3}/\overline{72} = \underline{3}/\overline{7 \text{ tens} + 2 \text{ units}}$$

$$\underline{6 \text{ tens}}$$

$$1 \text{ ten} + 2 \text{ units} \rightarrow 12 \text{ units}$$

III

Tens	Units
2	4
$\underline{3}/\overline{7}$	2 ←
6	+ 3
1 →	10 ←
	=
	12 ←
	12

IV

$$\begin{array}{r} 4 \\ 20 \\ \underline{3}/\overline{72} \\ \underline{60} \\ 12 \\ \underline{12} \end{array} \quad \text{or} \quad \begin{array}{r} 24 \\ \underline{3}/\overline{72} \\ \underline{60} \\ 12 \\ \underline{12} \end{array}$$

Some of the reasons why division is generally considered the most difficult of the fundamental operations with whole numbers are:

1. The algorism is difficult to rationalize.
2. Trial quotients are arrived at by estimation and often require corrections demanding insight beyond the ability of some pupils. In $46\overline{)322}$, for example, an estimated quotient of 8, obtained from $32 \div 4$, must be corrected. Likewise, 6 found from $32 \div 5$ is also in error.

3. Operations begin at the left and move to the right, contrary to the pattern learned for the other processes.

4. Division is the only operation with whole numbers in which all answers do not come out "even."

5. Zeros are required in quotients for examples such as $2\overline{)218}$ or $4\overline{)840}$. These are often omitted.

6. Multiplication of the divisor by each digit of the quotient is performed with the factors in positions different from those previously used. In the division $23\overline{)976}$, errors often arise in the multiplication of 23 by 4 because these factors are not located as they are in the conventional algorism

$$\begin{array}{r} 23 \\ \times 4 \\ \hline \end{array}$$

To overcome these difficulties, the teacher must know the meanings of the algorism as well as methods for effectively directing pupils in learning it. The desired end product is for all pupils to perform and apply the operation with understanding; yet only a few pupils can completely understand long division when the process is first introduced. Subsequent re-examination of the mathematical meanings, as well as refreshing and reteaching the algorism, are required for general understanding.[3]

Perhaps no one method of rationalizing long division has been proven best. It is also true that what is understood by one pupil may not be understood by another. For these reasons one should be acquainted with several methods used to illustrate the rationale of long division. A few techniques, explained briefly, follow.

Divide 19,826 by 23.

19,826 = 19 thousands + 8 hundreds

+ 2 tens + 6 units

There is no thousands of 23's in 19 thousands so 19,826 is decomposed to 198 hundreds + 2 tens + 6 units.

	H	T	U	
	8	6	2	
23/198	2	6		
184				Note:
14	2			↔ The 14 hundreds have been decomposed to 140 tens; plus 2 tens = 142 tens.
13	8			
	4	6		↔ The 4 tens have been decomposed to 40 units; plus 6 units = 46 units.
	4	6		

Divide 685 by 32.

32/685	
640	20
45	
32	1
13	21

Step 1. How many 32's might be in six hundred eighty-five? There could be 20 because $20 \times 32 = 640$.

Step 2. 45 remains. How many 32's in 45? Only one.

Step 3. Answer: $20 + 1 = 21$, with a remainder of 13.

Divide 8321 by 79.

8321 = 8 thousands + 3 hundreds

+ 2 tens + 1 unit,

but there are no 79 thousands in 8 thousands, so 8321 is decomposed to 83 hundreds + 2 tens + 1 unit.

	1 hundred + 0 tens + 5 units
79/83 hundreds +	2 tens + 1 unit
79 hundreds	
4 hundreds =	40 tens
	42 tens = 420 units
	421 units
	395 units
	6 units

105, R6.

Division algorithms were not always the same. Another algorithm which may interest teachers as well as exceptional students is the following:

Austrian Method:

Divide 17997 by 52.

346	Step 1.
52/17997	$179 \div 52 = 3$
239	$3 \times 2 = 6$, $6 + ? = 9$ (ans. 3)
317	$3 \times 5 = 15$, $15 + ? = 17$ (ans. 2)
5	
	Step 2.
	$239 \div 52 = 4$
	$4 \times 2 = 8$, $8 + ? = 9$ (ans. 1)
	$4 \times 5 = 20$, $20 + ? = 23$ (ans. 3)
	Step 3.
	$317 \div 52 = 6$
	$6 \times 2 = 12$, $12 + ? = 17$ (ans. 5)
	$6 \times 5 = 30$, $30 + ? = 30$ (ans. 0)

In conclusion one might say that for many, division is difficult. Through numerous experiences "infected" with under-

standing, it is hoped that one can discover and generalize the mathematical concepts of division. With guidance and with maturity one hopes for an understanding of the science of mathematics. Once this science is understood, the techniques for teaching it become increasingly easy to find and to use.

Notes

1. J. Houston Banks, *Learning and Teaching Arithmetic* (Boston: Allyn and Bacon, Inc., 1959), p. 17.
2. James A. McLellan and John Dewey, *The Psychology of Number* (New York: D. Appleton & Co., 1905), p. 220.
3. Marks, Purdy, and Kinney, *Teaching Arithmetic for Understanding* (New York: McGraw-Hill, 1958), pp. 176–77.

May it be known

The Board of Directors of the National Council of Teachers of Mathematics desires to encourage and support prospects that attempt to improve the teaching of mathematics in elementary schools.

Anyone who may have a good idea for the improvement of elementary-school mathematics and who is interested in obtaining financial support is invited to submit a complete description of his plans for implementing his idea. This proposal should be sent to the President, Phillip S. Jones, for consideration.

For an unusually promising project under qualified leadership, the council is prepared to give substantial support.

Did you know?

A bibliography of doctoral dissertations in elementary and secondary mathematics from 1918 to 1952 by Edward G. Summers appears in the May, 1961 issue of *School Science and Mathematics*. Two hundred and sixty-seven titles are listed under ten headings, including history and textbooks, teacher competencies, curriculum development, teaching methods, materials and instructional aids, problem-solving and understanding concepts, achievement, enrichment-remediation, evaluation, and teacher training. The list for the period 1951 to 1960 will appear in the next issue of the same journal.

A word of caution

"While striving to reduce the teacher load fairly, we must not deceive ourselves into assuming that there is a royal road to teaching any more than there is to learning. We should be eager to try new ways and yet not get so gadget happy that we can't face the limits as well as the possibilities of IBMetry."

—From *Education*, U.S.A., December 29, 1960

Understanding and the ability to solve problems[1]

ANGELA PACE *Syracuse University, Syracuse, New York*

Dr. Pace's dissertation won the award for the Delta Kappa Gamma Sorority single Fellowship for International Study. Thus, Dr. Pace not only is presently completing a year of post-doctoral study in England and in Italy but also is collecting data from some 6,000 children for purposes of making a study of comparative growth of basic mathematical understanding.

Most teachers will concede that improving ability to solve verbal problems in arithmetic is a difficult task. Children seem to have all manner of difficulties when it comes to making a decision about the process to use to solve a given problem. All too frequently they resort to guessing, to trial-and-error procedures, or to the use of verbal and number cues. The importance of number cues, for instance, is well illustrated by a child who, in a study made by Stevenson,[2] described her method of attacking problems in this way: "If there are lots of numbers I adds. If there are only two numbers with lots of parts I subtracts. But if there are just two numbers and one littler than the other, it is hard. I divides if they come out even, but if they don't I multiplies." As long as inefficient procedures like these are used, failure to solve problems is inevitable.

If children are to have any degree of success with verbal problems in arithmetic, it seems evident that they must have some understanding of the fundamental processes. They must understand not only the meaning of addition, subtraction, multiplication, and division, but also the effect of each of these when applied to quantitative situations. These understandings are a vital factor as far as successful problem-solving is concerned and systematic instruction in developing them should be provided.

In an effort to determine the effect of understanding of the processes upon problem-solving ability, a study was conducted in which two groups of children from the fourth grade of the Ella VanHoesen Training School were used. The two groups were equated with respect to three factors: chronological age, mental ability as measured by the California Test of Mental Ability, and average arithmetic scores obtained on the Arithmetic Reasoning and the Arithmetic Computation Tests of the Stanford Achievement Tests, Form J. One group was designated as the control group, or Group I; the second group, as the experimental group, or Group II.

Systematic instruction and its effect

The experimental period lasted for eight weeks. During this time, systematic instruction in understanding of the fundamental processes in arithmetic was provided for Group II only. Twenty-four especially prepared sets of problems were used as a basis for discussions, three sets being used each week. Each set contained, in random order, two one-step verbal problems which could be solved by addition; two, by subtraction; two, by multiplication; and two, by division. During periods of systematic instruction, children in Group II were asked to read the prob-

[1] This article is a brief, simplified summary of a part of a doctoral dissertation, *The Effect of Understanding on the Reorganization and Permanence of Learning—A Study in Verbal Problem Solving*, by Angela Pace.

[2] P. R. Stevenson, "Difficulties in Problem Solving," *Journal of Educational Research*, XI (February, 1925), 95–103.

Table 1
Comparison of gains made by Groups I and II from Form J to Form K of the Arithmetic Reasoning Test

Group	$\bar{X}$		t	P
	Form J	*Form K*		
I	3.37	3.43	.30	$>.70$
II	3.25	3.71	2.73	$<.02$

lems, to tell how they were to be solved, and then to defend their choice of process. In each case, the emphasis was upon *how* the problem was to be solved and *why* a given process was appropriate. Two questions asked constantly during the discussions were: "What do you need to do to solve the problem?" and "Why would you do that?"

According to plan, Group I was not provided with systematic instruction. During the experimental period, the children in this group merely solved the twenty-four sets of problems, identical to those used with Group II, but there was no discussion of the work.

In order to determine if the systematic instruction in developing understandings was of value, a comparison of gains made in problem-solving by both groups was necessary. For this purpose, Form K of the Arithmetic Reasoning Test of the Stanford Achievement Tests was given. A comparison of the results of Form J, which had been used as an initial measure, and of Form K was then made. Table 1 shows that only negligible gains were made by Group I. On the other hand, Group II made statistically significant gains ($P<.02$).

Problem Tests and interviews

In an effort to obtain added information relative to the growth of understandings and its effect upon ability to solve problems, four special tests were constructed for use during the experimental period. Each of these tests, devised to be as similar to each other as possible, consisted of fifteen verbal problems. In addition to ten problems of the conventional variety, each test contained five problems with distorted cues. These four problem tests—Problem Tests I, II, III, and IV—were given to Group I and Group II. Problem Test I was given during the week preceding the start of the experimental period. Problem Tests II, III, and IV were given during the third, sixth, and ninth weeks, respectively.

Because of interest in obtaining information regarding the development of understandings and its possible effect upon problem-solving ability, the administration of these tests was accompanied by individual interviews with the children. During the interviews, which were recorded on tape, the child was asked to state the process he would use to solve each problem and then to explain why he considered that particular process to be appropriate. In this way, data regarding not only the basis used by the child for selecting a process but also the relationship of that basis to success in problem-solving could be secured. It was felt that the interviews would help to provide answers to such crucial questions as these:

How do children solve problems?

Is a particular process selected because of the presence of certain word or number cues? Do children tend to "juggle the processes" to get an estimated answer? Do children make wild guesses? To what extent is the selection of process based upon an understanding of that process?

What effect does understanding have upon ability to solve problems?

Do children who understand the processes experience greater success than those who lack the requisite understandings?

Results of Problem Tests and interviews

In order to avoid too lengthy a report, only the results of Problem Test I, administered just prior to the beginning of the experimental period, and Problem Test IV, administered at the end of the

experimental period, are presented. The results of these two tests, together with an analysis of the accompanying interviews, provide data relative to growth of understandings during the eight-week period and its effect upon problem-solving ability.

Before the tables giving the results of the tests can be understood, however, an explanation of the tabulation of solutions and of children's responses is necessary.

First, the solutions to each of the problems on the problem tests were tabulated in one of four categories. In the category of *correct process* (CP) were listed those solutions which were entirely correct as to choice of process. In the category of *immature, correct procedure* (ICP) were listed all those procedures which were correct but which represented immature ways of solving problems. Examples of solutions tabulated as immature were:

1. 29 111111 for 35−29
2. 111/111/111/111/111/111/ for 36÷3
   111/111/111/111/111/111/

In the category of *incorrect process* (IP) were listed all solutions which involved the use either of an incorrect process or of a procedure which was not only immature but incorrect. Finally, procedures which were neither entirely correct nor entirely incorrect fell within a category labeled as *all others* (AO).

Next, the different responses made by the children, in recorded interviews, as to their reason for choosing a particular process to solve each of the problems on the problem tests were listed. Ten bases for choice of process, ranging all the way from mature understanding of processes to a complete lack of any understanding at all, were identified. An explanation of each of these follows.

1. Mature understanding. The child's choice of process was classified as being based upon mature understanding if he showed a clear grasp of the conditions of the problem and could give sufficient evidence that he understood why a given process was appropriate. Five examples of responses placed in this category are given. In each case, the statement of the problem is accompanied by the child's response indicating the reason given for choice of process.

Problem: Alice has 31 rabbits. Fifteen are big rabbits. The rest are baby rabbits. How many baby rabbits are there?

Child's response: "You know the whole group. You know part of the group and you want to subtract to find the other part."

Problem: Dick bought a new horn for his bicycle that cost $1.25 and a new basket that cost $1.19. How much did these 2 things cost together?

Child's response: "You add because the groups aren't the same size. You have to add because you're putting groups together."

Problem: Johnny bought 3 quarts of milk at 19¢ a quart. How much did 3 quarts of milk cost?

Child's response: "Multiply 'cause you're putting groups together. You know how much the milk cost. It cost 19¢ a quart and you know that Johnny bought 3 quarts of milk and you have to put three 19's together. It's a shorter way of adding."

Problem: Barbara has 84 postcards in her scrapbook. She can put 4 postcards on each page. How many pages will the 84 postcards fill?

Child's response: "If she could put 4 on each page, and she has 84, if you say how many groups of 4 postcards in 84 postcards, you'll come out with how many pages she can fill."

Problem: Mrs. White bought 3 pounds of cheese for $1.50. What was the price of 1 pound of cheese?

Child's response: "You have $1.50 and you've got to divide it up—break this group up into 3 different groups."

2. Immature understanding. The choice of process was classified as being based upon immature understanding if the child used a correct but immature procedure and was then able to show that he comprehended the conditions of the problem clearly. An example of a response classified in this category, together with the solution to the problem, follows:

   Solution:

   ~~IIIIIIIIIIIIIII~~IIIIIIIIIIIIIIII

   Child's response: "It's all about rabbits. I put 31 marks down instead of drawing pictures of rabbits. And it said 15 are big rabbits and the rest are baby rabbits. How many baby rabbits are there? So I took 15 marks away and then I counted the rest and I got 16."

3. Insufficient evidence of full understanding. The basis for choice of process was classified in this category if the child's explanation did not provide quite enough evidence that he really understood the process. Two examples of such responses follow:

   "Alice had 31 rabbits and 15 of them were big and it asks you how many baby rabbits there were."

   "Here it says he bought a couple of things for his bicycle. One cost $1.25 and the other cost $1.19, so I thought it must be adding."

4. Partial understanding. The child's response was classified in this category when only a partial grasp of the conditions of the problem was indicated. For example, in a problem involving division, a child's response was: "Well, she paid $1.20 for 3 and it said what was the price of one. You'd have to divide to get the answer." In this instance, the child recognized the need for division but her solution indicated only partial understanding. Her solution was:

$$\begin{array}{r} 0\ 20 \\ \hline 1)\$1.20 \end{array}$$

5. Incorrect grasp of the conditions of a problem. In this category were included responses which showed that the conditions of the problem had been misunderstood. In some cases, this was due to inability to read the problem. Responses indicating a lack of understanding of what happens when certain processes are used were also classified here. Three examples follow:

   "Well, they wanted to buy 4 things and I don't know what they are. I guess one of them cost 32¢ and they had to find out how much 4 of them would cost together." (In this instance, however, the problem asked for the price of one ticket when four cost 32¢.)

   "I subtracted 'cause you want to find out how much the two things cost together."

   "Lee bought 3 tennis balls for $1.80 and what was the price of one ball. They want to find out so you multiply."

6. No reason. Responses indicating inability to give an adequate explanation for choosing a process were tabulated in this category. Examples of such responses were:

   "I couldn't do it any other way."

   "Because that's the easiest way to do it."

7. Verbal cue. The response was listed here if there was evidence that a process was selected because of a key word or group of words in the problem, as in this example:

   "Well, it said, 'How many more pennies did Nancy have than Jean?' When it says *how many more*, you're supposed to subtract."

8. Number cue. In this category were listed responses, like the following, which indicated that a process was chosen because of the size of the numbers or the number of numbers in a problem.

   "There was 5, 29, and 5 so I added."

"I couldn't very well subtract on that one because there wasn't any number to subtract on."

"I added. There was a number on the top and bottom and one in the middle —a bigger one. I feel that I can't do it any other way."

9. Elimination. In this category were listed responses which showed that a particular process was selected after other processes had been eliminated. Examples follow:
"I was trying subtraction and it didn't work in that way. And then I tried adding. It didn't work that way so then I went to dividing and it sounded reasonable."

"Well, if I subtracted, I would get 74. And if I added, I'd get $1.86. So I divided."

10. Guessing. This category included responses in which it was evident that the choice of process was made in a purely random fashion. Examples follow:
"I didn't know what to do so I added."

"I just added because I wanted to."

Results of the Problem Tests

Table 2 shows the results of Problem Tests I and IV with respect to solutions to the conventional problems and to reasons for choosing the processes which were used.

Table 2 is read as follows: On Problem Test I, Group I had 31 solutions in which correct processes were used and 18 in which immature, correct procedures were used. Of these, 5 solutions were based upon understanding; 7, upon immature understanding; 20, upon insufficient evidence; 0, upon partial understanding; 5, upon no reason; 1, upon elimination; 1, upon guessing; 2, upon verbal cue; 0, upon number cue. The remainder of the test is read in a similar manner.

The table shows the following results:

1. Both groups showed improvement, from Problem Test I to Problem Test IV, in number of correct solutions used to solve the conventional problems. Group II showed greater improvement than Group I. Out of a possible total of 150, 32.7 per cent of the solutions offered by Group I on Problem Test I were correct. On Problem Test IV, 68.0 per cent of the solutions were correct. On Problem Test I, 30.0 per cent of the solutions offered by Group II were correct. On Problem Test IV, 84.7 per cent of the solutions were correct.
2. Both groups showed an increase in number of correct solutions based upon mature understanding and immature understanding. Group II showed a greater increase than Group I. Twelve correct solutions on Problem Test I and 32 correct solutions on Problem Test IV were based upon either mature or immature understandings in Group I. Twenty-eight correct solutions on Problem Test I and 88 correct solutions on Problem Test IV were based upon either mature or immature understandings in Group II.

Table 2
Results of Problem Tests I and IV with respect to the conventional problems

Group	*Problem Test*	*Solutions*		*Basis for correct solution*									
		CP	*ICP*	*U*	*ImU*	*Insufficient evidence*	*PU*	*Lack of understanding*					
								NR	*E*	*G*	*VC*	*IncG*	*NC*
I	I	31	18	5	7	20	0	5	9	1	2	0	0
I	IV	72	30	18	14	35	4	14	11	2	0	3	1
II	I	23	22	12	16	10	3	2	0	1	0	1	0
II	IV	123	4	87	1	33	2	1	0	2	0	1	0

Table 3
Results of Problem Tests I and IV with respect to problems with distorted cues

Group	Problem Test	Solution		Basis for correct solution									
								Lack of understanding					
		CP	ICP	U	ImU	Insufficient evidence	PU	NR	E	G	VC	IncG	NC
I	I	15	15	3	8	8	0	5	1	1	1	2	1
I	IV	31	14	9	10	11	1	9	2	1	0	2	0
II	I	16	13	6	10	7	0	2	0	4	0	0	0
II	IV	57	1	44	0	7	4	2	0	0	0	1	0

3. The increase in number of correct solutions for both groups was accompanied by an increase in number of correct solutions based upon mature and immature understandings.

In addition to the ten conventional problems, each of the problem tests also included five problems with distorted cues. These five problems were included to test the ability of the children to apply understandings of processes in solving problems of a slightly more complex nature.

Table 3 shows the results relative to the problems with distorted cues.

Table 3 shows these results:

1. Both groups showed improvement, from Problem Test I to Problem Test IV, in number of correct processes and procedures used to solve the problems with distorted cues. Group II showed greater improvement than Group I.
2. Both groups showed improvement in number of correct solutions based upon either mature or immature understandings. Greater improvement was made by Group II than by Group I.
3. The increase in number of correct solutions to problems with distorted cues was accompanied by an increase, for both groups, in the number of correct solutions based upon mature and immature understandings.

Postexperimental period

The experimental period was followed by another eight-week period during which Groups I and II received identical instruction in arithmetic which involved no solving of problems. At the end of this time, Form L of the Arithmetic Reasoning Test of the Stanford Achievement Tests was administered to determine if gains made in problem-solving during the experimental period were maintained.

Table 4
Comparison of results of Forms K and L of the Arithmetic Reasoning Test

Group	$\bar{X}$		t	P
	Form K	Form L		
I	3.43	3.87	2.19	$<.05$
II	3.71	4.43	5.42	$<.001$

Table 4 shows that Groups I and II not only maintained gains but improved considerably.

During this period, another problem test—Problem Test V—was also given and the results of this test were compared with those of Problem Test IV. Tables 5 and 6 show that both groups were able to maintain gains.

Implications

The improvement of problem-solving ability is not an easy task. This study has revealed information which can be of value as a means of increasing success with verbal problems.

First, the teacher needs to see that children are given many opportunities to solve problems. Children improve greatly if they are merely given many problems to solve and then left to their own devices in solv-

Table 5
Results of Problem Tests IV and V with respect to the conventional problems

Group	Problem Test	Solutions		Basis for correct solution									
								Lack of understanding					
		CP	ICP	U	ImU	Insufficient evidence	PU	NR	E	G	VC	IncG	NC
I	IV	72	30	18	14	35	4	14	11	2	3	1	0
I	V	90	25	29	12	35	2	19	14	3	0	1	0
II	IV	123	4	87	1	33	2	1	0	2	0	1	0
II	V	124	8	82	8	32	2	1	3	2	0	2	0

Table 6
Results of Problem Tests IV and V with respect to problems with distorted cues

Group	Problem Test	Solution		Basis for correct solution									
								Lack of understanding					
		CP	ICP	U	ImU	Insufficient evidence	PU	NR	E	G	VC	IncG	NC
I	IV	31	14	9	10	11	1	9	2	1	0	2	0
I	V	37	9	11	9	10	2	8	2	1	0	3	0
II	IV	57	1	44	0	7	4	2	0	0	0	1	0
II	V	57	4	40	4	12	3	2	0	0	0	0	0

ing them. It seems evident that, if children are to have greater success with problems, they must be provided many opportunities to work with problems.

Secondly, children should be allowed to solve problems in various ways. Teachers sometimes tend to hinder the problem-solving process by insisting upon the use of one correct procedure. There is more than one correct way of solving a problem. This study showed that some children used a variety of correct but immature procedures. Children should be free to use immature procedures but should be led gradually to use more mature and efficient processes.

Lastly, the teacher should provide for the development of understandings of the four fundamental processes, since this is a vital factor as far as improvement in problem-solving is concerned. The child needs to know what happens when each of the processes is used. He needs to understand *why* he adds, *why* he subtracts, *why* he multiplies, or *why* he divides. He needs to understand that addition and multiplication are both "combining" processes, that he adds when two or more groups of unequal size need to be put together, and that he multiplies when the groups to be combined are equal in size. Furthermore, he needs to understand that subtraction is a "separating" process and that it is essentially the process used to find the other part when the total and one of the groups are given. He needs to know that division is also a "separating" process and is used for problems where it is necessary to separate the total group, either (*a*) to find the size of a given number of groups into which this total group has been separated, or (*b*) to find the number of groups of a certain size into which the total group can be separated. Understandings such as these are essential for success in problem-solving.

The development of understandings is a gradual process, however. Understandings cannot be given to children as rules to be committed to memory. They must be developed as the fundamental processes are presented. They must be developed fur-

ther as problems are solved. Children should be asked constantly to explain how a problem is to be solved and why a particular process is thought to be appropriate. This procedure tends to prevent pure guessing and the random manipulation of numbers, and it forces the children to think about the meaning of the processes they are using in solving problems.

Children show gains in problem-solving ability if they are merely presented with many problems to solve, but they show even greater gains if systematic instruction for the purpose of developing understandings of the four processes is provided by the teacher.

Teacher exchange program opportunities, 1962-63

Once again it is time for teachers who would like the experience of teaching in a national or in an American-sponsored school in another country to begin planning for a year abroad. This unusual opportunity is offered to elementary, secondary, and junior college teachers in all subject fields, through the teacher exchange program administered by the Office of Education in co-operation with the Department of State as authorized by Public Law 584, 79th Congress, the Fulbright Act, and Public Law 402, 80th Congress, the Smith-Mundt Act.

During the 1962–63 school year, approximately four hundred grants will be available in forty countries. The exchange teaching assignments are of two types: those involving an interchange of teachers and those involving a one-way assignment for an American teacher in a school abroad. Summer seminar grants for teachers of French, German, Spanish, Latin, and history will also be available in France, Germany, Colombia, and Italy, and in Brazil and India, respectively.

The basic qualifications for an exchange teacher include the following: United States citizenship, a bachelor's degree, three years of successful teaching experience, good physical health, moral character, emotional stability, maturity, and adaptability. Other considerations being equal, veterans and persons under fifty years of age are given preference.

Being a well-qualified teacher is not enough. A candidate should have a real interest in the culture, history, educational system, and people of the country for which application is made. The teacher should also have the ability and willingness to interpret the United States—its people, customs, and culture—to people abroad.

Applications and complete information regarding teaching and summer seminar opportunities for the 1962–63 academic year can be obtained after August 1 from the Teacher Exchange Section, Office of Education, Department of Health, Education, and Welfare, Washington 25, D.C. Applications will be received until October 15, 1961.

On the fraction as a numeral

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In the December, 1960, issue of THE ARITHMETIC TEACHER, there is presented a rather cogent case for the use of rate pairs and the definition of a fraction as a number.*

It is the purpose of this article to discuss a key aspect of a different system of ideas: the fraction as a symbol, and with it some consequent implications, particularly in the area of problem-solving.

Of basic importance in our discussion will be the distinction between an object and a name for that object. In more familiar contexts this distinction causes little difficulty; for example, few would confuse the four-legged animal he can actually pat, curry, or ride with its symbolic, five-lettered English referent: *h-o-r-s-e*.

In the past few years a great deal of attention has been given to the distinction between a number (a concept) and its numeral (a symbol, conventional or otherwise, which stands for the number concept).

With respect to fractions, the solution to the equation

$$2x = 1$$

is but a single number (not a family of numbers) that may be referred to variously in language as *one-half* (English), *demi* (French), *medio* (Spanish), etc., or in symbols by a large "family" of accepted numerals: $\frac{1}{2}$, .5, $\frac{4}{8}$, $\frac{1}{4}+\frac{1}{4}$, 50%, etc.

More precisely, for us a fraction is but a mathematical symbol—a numeral—consisting of three parts: two numerals and a bar between them. The numeral above the bar, or to the left of it, we call the *numerator;* the numeral below the bar, or to the right of it, we call the *denominator.* (A similar symbol in nonmathematical situations, of course, may not be a fraction.) It will be useful for subsequent mathematics to broaden this definition to include numerators and demoninators which are variables, i.e., symbols that may be replaced by numerals. The definition then covers not only the simpler fractions of arithmetic, but the more complex ones of later mathematics as well; e.g.,

$$\frac{\pi}{6}, \quad \frac{\sin\alpha}{\cos\beta}, \quad \frac{\log 3}{e}, \quad \frac{8+\dfrac{x}{7}}{\sin x}, \quad \frac{\sum_{i=0}^{10} x_i y_i}{D_x y}.$$

In each case we are dealing with mutually agreed upon symbols for certain mathematical entities, or "numbers." And, parenthetically, no matter how complex they may appear within standard mathematical contexts, they adhere to the same computational pattern that applies to the simpler fractions of arithmetic.

One consequence of this definition of a fraction as a symbol, though, is the logical need to strike from our vocabulary a now-inconsistent term: "decimal fraction." The expressions .6, .36, 1.42, in our context, are but a different set of numerals —an alternate set of number expressions— representing the same *numbers* as the symbols: $\frac{3}{5}$, $\frac{9}{25}$, $1\frac{21}{50}$, respectively. This is analogous to the language situation in which the word-symbols *red*, *rouge*, and

* Henry Van Engen, "Rate Pairs, Fractions, and Rational Numbers," THE ARITHMETIC TEACHER, VII (December, 1960), 389–99.

rojo all stand for the same color, but are appropriate in the different language contexts of English, French, and Spanish, respectively. If one will review the historic beginnings of decimals, he will note that Simon Stevin, who is considered their inventor, developed decimals as an alternate set of symbols to avoid the computation complexities inherent in the use of fraction symbols.

Since there is no numerator or denominator to decimal numerals, it is inaccurate, by our definition, to refer to them as "decimal fractions." Numerals of this style expressing numbers both greater and less than one, e.g., 1492, 1.62, and .04, may be referred to as decimals, short for "decimal expressions."

This particular change in terminology represents no great loss. For one thing, it does away with the usual but not-too-convincing rationalization that "in decimal fractions the denominator is not written (or is understood)." In decimal expressions there is no denominator; denominators are characteristic of fractions (symbols) only.

Finally, in this fraction-as-a-numeral scheme, per cent expressions are regarded as belonging to still another set of symbols. Experience clearly indicates that in certain situations and circumstances we find it convenient to express our number concepts, symbolically, in different ways. For example, if one wishes to communicate to another that one out of every two pupils in a class is under age 12, he may use any one of three classes of number expressions, typified by $\frac{1}{2}$, .5, or 50%. To show that 3 of 4 potted plants are blooming he may use, $\frac{3}{4}$, .75, or 75%.

Equivalence sets

Of fundamental importance in any generic development of fractions is the matter of deciding whether or not two fractions represent the same number. We accept, as one of the basic properties of fractions, the following "cross-product" criterion:

$\frac{a}{b}$ and $\frac{c}{d}$ represent the same number $\left(\text{or } \frac{a}{b}=\frac{c}{d}\right)$ if, and only if, $a\times d=b\times c$ (zero denominators expected).

For example, $\frac{6}{21}$ and $\frac{10}{35}$ are fractions representing the same number, since $6\times35=21\times10=210$. Fractions for which this criterion holds belong to the same equivalence set. Thus $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \cdots\}$ is but a set of equivalent numerals for a single number, since for any pair in the set the cross-product criterion holds.

But, a far more crucial point raised by our cross-product criterion is the clear implication that we intend to multiply symbols.

Thus, we meet head-on the question: Do we operate on symbols, symbols and numbers, or numbers?

Generally, it has been accepted that we operate only on numbers. But let's explore this position a bit. Accepting the distinction between number and numeral implies that we distinguish between these two orders: the conceptual realm of number and the symbolic realm of numeral. Computation with pencil and paper, then, is essentially an operation on numerals—the representations of numbers—and therefore belongs to the realm of numeral. The pencil-and-paper feature is not limiting: the same would hold for chalkboard and chalk or the operations of a high-speed computing machine. Moreover, the implications of this stand far exceed the realm of fraction notation, so we may illustrate with integers.

Consider this situation: A new outboard motor sells for $381. You know you have $85 in a bureau drawer, and you are due a return of $78 on your withheld income tax. For how much, you wonder, will you have to nip the bank account in order to buy that motor?

You go to work on the back of an envelope with a pencil. You put down:

$$\begin{array}{r} 85 \\ \underline{78} \end{array}$$

and add: "8 and 5 is 13 . . . (write 3) . . . carry 1 . . . 7 and 8 and 1 is 16 . . . (write 16) . . . that's 163." Now, you write:

$$\begin{array}{r} 381 \\ 163 \\ \hline \end{array}$$

and subtract: "3 from 11 is 8 . . . (write 8) . . . 6 from 7 is 1 . . . (write 1) . . . 1 from 3 is 2 . . . (write 2) . . . ." Surveying your piecemeal handiwork you read the last symbol and muse, "218 dollars—gosh!"

What was going on on the back of that envelope? Granted you were trying to find the sum of two numbers—78 and 85—and a moment later the difference of two numbers—381 and 163—but what really took the time was your work with numerals.

No serious problems are foreseen in declaring that you first added two numerals and then subtracted two numerals. There seems to be no overriding merit to reserving the language terms *add, subtract, multiply,* and *divide* for operations on numbers only and denying their use in "figuring" contexts.

As a compromise, we might elect to use single quotes, such as 'add,' 'subtract,' 'multiply,' and 'divide,' when using these terms in connection with "processing" numerals, or whatever it is we do as we compute answers. There was a great deal of this sort of thing a few years ago when our attention was first directed toward the number-numeral distinction, but we have tended to outgrow it and let the usually obvious context take care of the distinction. We have been doing this for some time now in written language: we don't insist on phonetic marks for the word *wind,* say, when in our sentence we want to assure its meaning "moving air" rather than "meandering motion," or vice versa.

Granted the use of these operational terms in connection with both numbers and numerals (or with whatever appropriate marking you wish, should you feel the distinction *must* be made), it becomes perfectly valid to do such things as to add numerators of fractions, i.e., add symbols; to multiply denominators (symbols); and so on.

Actually, we have always said and done as much in arithmetic—carrying ones, adding by columns, etc.—before we began to pay much heed to distinguishing between number and numeral. It is contended here that the basic rationale suggested in this article offers considerable prospect for putting long-standing computational thought patterns on more modern, firm, and honest ground.

Problem-solving

Unquestionably, the system of ideas of which the fraction as a numeral is a part has considerable implications for problem-solving, the almost universal end for which arithmetic is studied. Here too we will find it useful to draw distinctions.

Let us suppose that there are some oranges in two different baskets. The problem: How many are there altogether? At the outset, the problem is clearly in the realm of oranges. We can move from that realm to the realm of number by applying some suitable unit of measure. Should we choose a single orange as the unit, we find that there are 18 oranges in one basket and 16 in the other. To find out, computationally, how many oranges there are altogether, we write

$$\begin{array}{r} 18 \\ 16 \\ \hline \end{array}$$

and add the symbols, column by column, getting for a result the symbol 34. This symbol represents a number which, in the realm of the problem, indicates to us that there are 34 oranges, collectively, in the two baskets.

But note the important effect of our choice of unit. Had we chosen the unit to be dozen, then the resulting numbers would have been very different: $1\frac{1}{2}$ and $1\frac{1}{3}$ instead of 18 and 16. Naturally, things would also have been quite different in the computation stage:

$$1\tfrac{1}{2} = 1\tfrac{3}{6}$$
$$1\tfrac{1}{3} = 1\tfrac{2}{6}$$
$$2\tfrac{5}{6} \cdot$$

The resulting symbol, $2\frac{5}{6}$, also indicates to us how many oranges there are altogether in the two baskets, and is just as accurate an answer as the previous 34. Yet, both the number stage and the numeral stage in the solution of this problem have been quite different, all because of a different initial—and quite arbitrary—choice of unit.

To emphasize another aspect of the role of the numeral in problem-solving, consider the computation example:

$$\begin{array}{r} 24 \\ -6 \\ \hline 18 \cdot \end{array}$$

Here the numerals could, with equal effectiveness, represent numbers of elemental units which might range in the physical realm from aardvarks to zygotes. Not only that, but the same computation might also serve a variety of essentially different problem situations. For instance: (*a*) How many apples would be left if 6 apples were removed from a box of 24 apples? (*b*) John has 24 apples and his sister Sue has 6. How many more apples has John than Sue? Both (*a*) and (*b*) in their resolution might use the same computational model:

$$\begin{array}{r} 24 \\ -6 \\ \hline 18 \cdot \end{array}$$

Yet the situations of these two problems are fundamentally different.

In the first instance, (*a*), we have a "remainder" situation: the subtrahend represents a quantity to be removed from the minuend quantity. More generally, the subtrahend represents a subset of the minuend set, and the remainder another subset of the minuend set, or that set complementary to the set represented by the subtrahend. In the second case, (*b*), we have a comparison situation. The sets represented by the minuend and subtrahend are disjoint (distinctly different, though matchable, of course) and are to be compared. The remainder represents an expression of that comparison, the excess subset of the minuend set when the elements of the subtrahend set have been paired in one-to-one correspondence with the elements of the minuend set. Clearly, then, we employ in these two familiar, but essentially different, problem situations the same computational model and rules.

Similarly, the fraction $\frac{2}{3}$ can have equal effectiveness as a representative symbol for the essentially different interpretations 2 "thirds," $\frac{1}{3}$ of 2, or the ratio 2 to 3. And this is more a convenience than a bother, for no matter what we use the symbol to represent for us in our problem situation, its essential activity in computation—an activity with symbols—is unchanged. Since our fraction rules of operation are sufficient to serve as a consistent computational model for practically all problems of ratio, there is no need in our context to distinguish between operations on rate pairs and operations on fractions. Furthermore, because special circumstances do exist in which the fraction symbol can be representative, but for which the usual computational model is invalid (e.g., in some cases the sum of two ratio fractions is not consistent with reality), is not a crucial matter. After all, $2+3=5$ does not serve for finding the sum of electrical resistances of 2 ohms and 3 ohms when connected in parallel, nor does it fit the quantitative result of adding 2 cu. ft. of sand to 3 cu. ft. of gravel. When the mathematical shoe fits the situation in reality, it may be worn properly and with comfort. No tool has universal applicability.

One further point: At the end of computational activity a resultant numeral emerges, which in itself is also completely devoid of any inherent problem meaning—until we attach problem significance to it.

It is important to recognize that computation, with its own peculiar structure, rules, and activities, has no *necessary* reference to reality. The structure of arithmetic—in fact, of all mathematics—may be looked upon as a very useful, though incidental, analog to nature. Put another way, in a physical world completely different from our present one, the man-made structure of arithmetic could exist with its integrity and mathematical validity undiminished. The mathematics of our present world may not prove to be as useful an instrument in that different world, but that would be of no internal consequence.

Four-stage activity

In summary, it is suggested here that problem-solving by arithmetic is essentially a four-stage activity:

I. In the realm of the problem: We interpret and analyze the problem—particularly its numerical aspects—decide on some operational course of action, and decide on unit(s) of measure.

II. In the realm of number: We apply the selected unit(s) of measure, and convert the problem data to numbers.

III. In the realm of computation: We express the numbers of Stage II as numerals and undertake computation according to the rules of our selected operation; the ultimate result is a numeral or numerals.

IV. In the realm of the problem: We interpret the resulting numeral(s) of Stage III in terms of our problem situation.

A few observations on this four-stage sequence may be pertinent.

1. In so-called one-step problems, this sequence is likely to be quite orderly; in "multi-stepped" problems there is likely to be considerable fluctuation between adjacent stages.
2. There is an interesting correspondence between these four stages and the role of man and the machine in problem-solving. Stages I, II, and IV operate essentially at the conceptual level, and therefore cannot be handled by machine; Stage III does not, so it could be handled by machines.
3. Though activities at Stage III are potentially replaceable by a machine, until such time as this is universally so, computation *per se* must be learned as an absolutely necessary sub-operation to arithmetic problem-solving. Toward this end, it is *at times* both defensible and economical to study computation for its own sake, divorced from specific details of reality.

Newly elected officers

The following officers and directors assumed the responsibilities of their respective offices at the annual meeting of the council at Chicago in April:

Vice-President for College: Bruce E. Meserve
Vice-President for Junior High School: Mildred Keiffer
Directors: William T. Guy, Jr.
Max Beberman
Julius H. Hlavaty

A spark of enthusiasm

—a report of a teaching procedure in fifth-grade arithmetic, which developed not only understanding but also interest and enthusiasm

LUCILE B. POWELL *North Tonawanda, New York*

Editor's note

Some projects are carried on which are inspirational to the teacher. Other projects prove interesting and valuable to students. Mrs. Powell shares with us a procedure which seemed very worth while and inspirational both to her and her fifth-grade children, a class of average ability—twelve boys and fourteen girls.

For some time, my fifth-grade classes have felt that arithmetic was a very dull subject. There is need for repetition and drill on the same work children have studied in the second, third, and fourth grades. Of course, pupils need, in varying degrees, this drill and review. Because of a deep interest and enthusiasm for mathematics, I wanted to convey this feeling to my fifth-grade pupils. An elementary teacher feels a responsibility to cover the required material in the textbook, for she does not consider herself qualified to select the material that may be omitted. Thus, I began to use a variation of the "contract type" teaching plan. This plan has been used for three years, changing and adapting to meet the needs of the class and the individual pupils within the class.

Procedure

At the beginning of the school year, the class works together, exploring the work required for the year. Teacher prepares tests, assigns work, and standardizes achievement tests. Pupil and parent conferences are used to gain knowledge of the ability and interest of each child in the class. After about two weeks, we are ready to begin the plan whereby children may work as rapidly as they wish in arithmetic.

Pupil responsibility. Each child is expected to complete and to hand in a minimum number of assignments, preferably at least one a day. He indicates on a master record-sheet each time he completes an assignment. The pupils check this sheet many times a week to see how many assignments they have handed in or how many they are lacking. These assignments are corrected and returned to the pupil as they are completed. Corrections are then made and the papers are returned. If the assignment has a grade of C it must be corrected and returned. If the grade is below C, the entire assignment must be done again. In both cases, the papers are rechecked and a revised evaluation made of the work.

Teacher responsibility. I accept the responsibility of correcting the papers and recording the results. The children know that their papers will be corrected and handed back to them. They are disappointed if the papers are not ready each morning.

Each day we have a planned lesson-discussion in which the entire class participates. The topic may be some work in which the class needs help, or the development of a new idea. Following this class discussion, each child works on his assigned work. The period provides opportunity to do individual work with several

children. If three or four have the same difficulty, we have a small group working together. Also, time is used for individual help when only a part of the class is in the room, as during chorus, band, or religious instruction.

After the majority of the class has completed a unit, we have a testing and evaluation of each child's work. If a child has need of more work in a particular area, his assignments are adjusted to include this work. Although his progress may be delayed, he has a better understanding of arithmetic ideas.

If children wish to work "ahead" of class discussion, they study the several trial examples and problems provided for this purpose in the textbook. This reading in arithmetic is of a different type than that in the regular reading class. The child must understand what he is reading and must be able to apply this understanding to the work he must complete. This reading is not for speed or skimming, but for greater comprehension. This also aids the child in his interpretation and understanding in problem study. The child who understands what he reads has no fear or dread of a page of problems. Some competition with other members of the class or with special groups in the class is inevitable. This is no less or greater in extent than exists in any class.

Children know whom to ask for help. They know which pupils can give them the best help. A great deal of student help is neither encouraged or discouraged. However, an effort is made to see that both children—the pupil and the pupil's teacher—benefit from these experiences. Many children learn better from their peers than they do from their teacher. However, children are encouraged to come to me when they need help.

Often parents will help the children. We have found no conflict in this method. Some children need individual help, and more help than the teacher has time to give. If there is understanding, the method makes little difference.

Results

We have found no evidence of copying papers or sharing papers already marked. No child does all his work at home. This enables one to supervise the work he does in the classroom and to observe his work habits; there is a relaxed atmosphere when the class is working. The children are allowed to work on arithmetic at various times during the day. Thus, they do not feel forced to finish their work by a predetermined time. Some children choose to work at noon and some even stay after school.

The groups are eager for new work. We have introduced some algebra, using unknowns, and they became very excited because they could do some work like that which their older brothers and sisters were doing. We watched an educational television program on "The Algebra of Sets." We tried some equation-solving using the terms: *sets, universe, replacement sets.* They did surprisingly well and have asked for more.

Members of the class have made their own individual graphs of their week's marks and averages. They have been free to choose the kind of graph they wish to make. They study their marks as they graph them, and realize how well they are achieving in arithmetic. Parents, too, have been able to watch their children's progress from the graphs. During one marking period, I first evaluated the children's work. Then, I asked them to estimate what they felt they deserved for a mark, using their graphs as a basis for their estimate. There was an unbelievable amount of similarity.

Pupil reaction. The following are quotes made by members of the class: "I'll work at home and get a lot done"; "I do not have any time when I have nothing to do. I can always do arithmetic" (after we read about leisure time in the *Weekly Reader*); "I had trouble with that page once. Now let me help him with it"; "I'm going to finish my book by the end of the year. I have it all planned."

In response to some questions asked by the principal of the school about their work in arithmetic, the results were as follows:

1. Have you noticed any difference in the way arithmetic was taught this year as compared to other years? Yes—23.
2. Do you feel that you have any more, the same, or less understanding because of this different way? More—19; same —4.
3. Do you feel that you like arithmetic more, the same, or less than when taught the other way? More—19; same —4. The word "fun" was used 4 times; the word "wonderful" was used once to describe their feeling.
4. Could you suggest a change in the way it was taught that would have made you like it more or understand it better? More time for arithmetic—6; more homework—6; longer time for class—6.

Answers to other questions asked of the class as a group are as follows (25 children were present):

1. How many take home all the arithmetic papers after they have been marked by the teacher? 15
2. How many discuss their arithmetic papers with their parents? 18
3. How many take home only A or B papers? 3
4. How many have received help from their parents in arithmetic this year? Lot of times? 4 Once in a while? 20
5. How many have received help from older brothers and sisters lots of times? 1 Once in a while? 8
6. How many feel they understand the arithmetic they are studying? 21
7. How many like to do arithmetic? 19

Some children have grown in understanding to such a degree that they have put themselves in the above-average group at the end of the fourth marking period. To me, if seems that we have more interest, enthusiasm, and understanding in arithmetic class than I ever thought possible with fifth-grade pupils.

Are you planning to submit a manuscript?

We invite you to write for publication in THE ARITHMETIC TEACHER. We ask, however, that you prepare your manuscript according to the following instructions:

1. All manuscripts should be typewritten, double- or triple-spaced, on 8½ by 11 good quality paper, with *at least* one-inch margins on both sides.

2. References and footnotes should be numbered consecutively and placed at bottom of the page.

3. Prepare footnote references as follows:

To a book:

[1] Francis J. Mueller, *Arithmetic, Its Structure and Concepts* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1956), p. 17.

To an article in a periodical:

[2] Marshall H. Stone, "Fundamental Issues in the Teaching of Elementary School Mathematics," THE ARITHMETIC TEACHER, VI (October, 1959), 179.

To a yearbook:

[3] Robert L. Swain, "Modern Mathematics and School Arithmetic," in *Instruction in Arithmetic*, Twenty-fifth Yearbook of The National Council of Teachers of Mathematics (Washington, D.C.: The National Council, 1960).

To a technical bulletin, pamphlet, or similar publication:

[4] *Studies in Mathematics Education* (Chicago: Scott, Foresman and Company, 1960), p. 22.

4. If a bibliography is included at the end of the article, use the following guide.

For a book:

SPITZER, HERBERT F. *The Teaching of Arithmetic*. Boston: Houghton Mifflin Company, 1954.

(*Continued on page 250*)

Number systems, fad or foundation?

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Inventing sets of names and symbols for the purpose of exploring number systems and notation by analogy has become quite common. Since there seems to be little or no evidence of the effect the use of these systems has on students, the following report is presented to document the effect which can be expected when such material is used on collegiate teacher trainees under the conditions described.

The report consists of two parts: first, a summary of results in a controlled experiment conducted in 1955 and 1956 at Iowa State Teachers College and second, a summary of questionnaire data obtained from teachers who had used such invented material when they were in school. A quote from the original report of this project will explain the origin and intent of the material used in the experiment.[1]

> The materials referred to as "Make-Believe Arithmetic" consist of a set of unique names and symbols for sets up to six. It uses a symbol for the empty set and position to make an arithmetic analogous to that in common use. The number names consist of vowels with the long sound together with an initial "l." Simple combinations of bars and curves make the symbols and each can be written with an unbroken stroke of the pencil or chalk. The choice of the vowel sounds was made so as to enable the student to learn rote counting almost immediately. Authorities differ as to the theoretical value of the child's being able to count by rote, but nobody denies that most children have some ability to do so when we begin to work with them at school. Therefore, it seemed advisable to put the student learner in the same position. The base of six reduces the load of combinations which must be *brought under control* before the student can compute well. Six is a triangular number which makes many of the visual devices and diagrams work out very similarly to their base ten analogy. Also, counting by pairs and by half the base goes very well using base six.

[1] Elbert W. Hamilton, "The Notational System as an Aid to Understanding Arithmetic" (Doctor's dissertation, the State University of Iowa, 1956).

The idea for developing this analogy grew out of dissatisfaction with the usual exercises proposed in the books used in teacher education. Students failed to really understand the notion of base. If exercises were in base ten, they knew it so well that they weren't challenged to really examine the operation, and if exercises were in another base, they got so confused over the different meaning attached to the Arabic symbols that they couldn't do anything. The answer seemed to be a complete analogy, simple enough to admit of easy learning but capable of displaying most of the characteristics of base ten arithmetic. An added dividend, of course, was the opportunity to study learning to count.

The essential features of the system used follow:

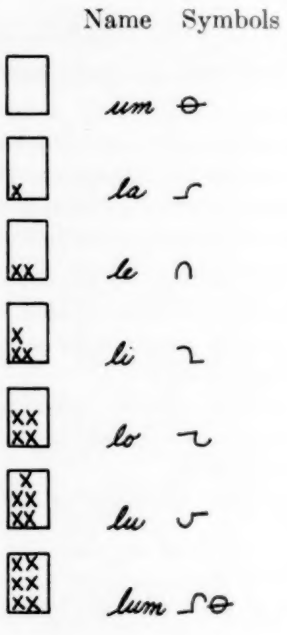


Figure 1

The place value chart looked like this:

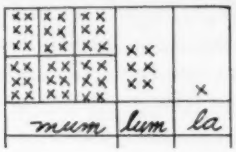


Figure 2

An addition table looked like this:

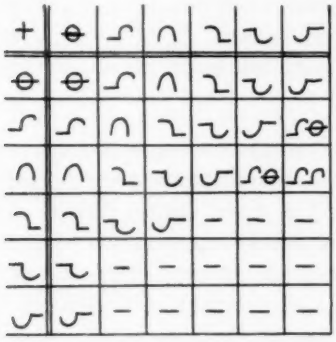


Figure 3

A multiplication table looked like this:

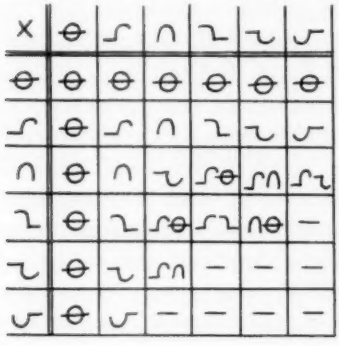
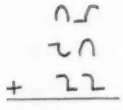


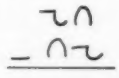
Figure 4

Typical exercises looked like the following:

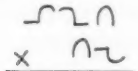
Addition:



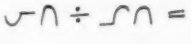
Subtraction:



Multiplication:



Division:



Fraction charts were introduced; ordinary fraction notation was as follows:

Positional fraction notation (base six) came from extending the place value chart (Fig. 5).

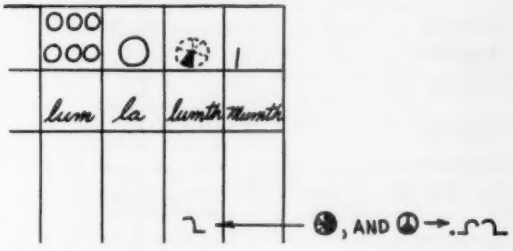


Figure 5

A ruler was also constructed, and activities in measurement were carried out in the system.

Enumeration (words)	Enumeration (symbols)
la	
le	
li	
lo	
lu	
LUM	
lalum	
lelum	
lilum	
lolum	
lulum	
LESY	
lesy-la	
lesy-le	
lesy-li	
lesy-lo	
lesy-lu	
LISY	
lisy-la	
lisy-le	
lisy-li	
lisy-lo	
lisy-lu	
LOSY	
losy-la	
losy-le	
losy-li	

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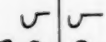
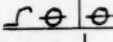
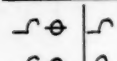
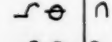
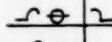
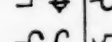
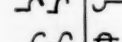
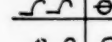
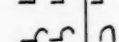
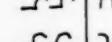
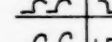
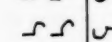
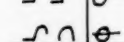
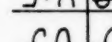
lusy-lu	
la MUM	
la mum la	
la mum le	
la mum li	
la mum lo	
la mum lu	
la MUM LUM	
la mum lalum	
la mum lelum	
la mum lilum	
la mum lolum	
la mum lulum	
la MUM LESY	
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Ordinals	*Fractional parts*
lath	leth
leth	lith
lith	loth
loth	luth
luth	lumth

Simple measures:

thumb
cubit
lelum thumbs equal la cubit

About twenty-five pages of illustrations, questions, and exercises led the students to learn to count, write two- and three-place numbers, add, subtract, multiply, and divide with whole numbers and fractions. The abacus, lum blocks, pocket chart, counting rod, and flannel board with appropriate material in base six were found to be almost indispensable, for students really had to learn to perform and were tested on their ability to do so.

The criterion test used in the research study, however, consisted of two forms of a forty-seven-item multiple-choice test over regular base ten arithmetic. The clusters of items contained in the test can be classified under the following headings:

Notation and base	7 items
Positional values	8 items
Performance in computation	11 items
Generalization	6 items
Rationalizing operations	11 items
Ratio	4 items

All sections of a beginning course in the teaching of arithmetic were tested on the first day of class. Intact classes of thirty-three or thirty-four students were assigned at random to three experienced teachers in such a manner that each teacher had an experimental section which was to use the "Make-Believe" material and a control section which was to use traditional number-system material for specified blocks of time. The control groups averaged 21.51 on the test as a prescore, and the experimental groups averaged 21.55, so no further matching was attempted. At the end of three weeks, all students took the alternate form of the test, and at the end of twelve weeks, all students took the same form they had taken in the beginning. The results of instruction were judged on the basis of gain scores at the end of the three-week and at the end of the twelve-week periods. Eighteen tables were compiled; seven of them were analysis of variance tests of the difference in gain by experimental groups using the "Make-Believe Arithmetic" to supplement the regular text as compared to control groups using the more traditional number systems suggested in all the texts available at that date.

A summary of the results of these tests is quoted from the original report:[2]

> The material called "Make-Believe Arithmetic" proved to be more effective than conventional textbook material in producing understanding of place values and number systems when used on teachers, college freshmen,

[2] *Op. cit.*

and sophomores. Three comparisons were made. The first involved a three-week period in the fall of 1955. During that time a control group studied only conventional material and an experimental group studied "Make-Believe Arithmetic." The results favored the experimental material but the difference was not significant. The second comparison made during the winter quarter for another three-week period favored the experimental material and the difference was significant beyond the 5 per cent level of confidence. The third comparison was between the fall experimental group and a control maintained for twelve weeks during the winter quarter. The result favored the experimental material and the difference was significant beyond the 5 per cent level of confidence.

1. The material called "Make-Believe Arithmetic" produced better results when used at the beginning of a term as a concentrated attack upon the number system than when added to conventional material later in the term. The difference was not significant at the 5 per cent level.
2. The material called "Make-Believe Arithmetic" appears not to favor any particular ability group. No interaction for levels was significant at the 5 per cent level.
3. The material called "Make-Believe Arithmetic" appears not to favor any particular teachers. No interaction for teachers was significant at the 5 per cent level. All three of the research comparisons favor the experimental material. In two of the three comparisons the difference was significant at the 5 per cent level of confidence.

This material had been in use long enough to make it possible to question a sample of teachers in the field who had up to four years' experience and who had had "Make-Believe Arithmetic" as part of their training.

A cross-section sample was drawn, and returns, although limited to 270 respondents, fitted the sample very well. It was possible to categorize the anonymous responses as to years of experience, grade level of assignment, and grade received in course, and responses were elicited to twenty-eight questions. Most of these were qualified responses on a scale, but a few were categorical yes, no, or required written explanations.

Again, the conclusions quoted from the original document give the best account of the results obtained:[3]

[3] *Op. cit.*

1. The questionnaire showed the material called "Make-Believe Arithmetic" to be very popular with teachers who had experienced it as part of their training. Eighty-six per cent recommended continued use of it.
2. Of those responding in favor of continuing the use of the material, 32 per cent recommended more use than they had enjoyed. Fifty-three per cent favored the same use they had experienced and 14 per cent recommended less use of it.
3. The strongest expressions about the material concerned understandings, 99 per cent favorable; the nature of counting, 93 per cent favorable; place value, 92 per cent favorable; and visual aids, 92 per cent favorable.
4. Very strong professions of confidence, especially with regard to subject matter, were given by the respondents.

Eighteen pages of favorable free comments, typed single space, were transcribed from the questionnaires. Only two pages of unfavorable comments were found, and many of them were pointless complaints about present difficulties or the rigidity of the school systems in which they were teaching. On the basis of this evidence, it would appear that time spent by prospective teachers in exploring number systems is time well spent. No evidence is presented for or against such exploration on the part of grade school and high school students.

Certainly, no case can be made for these particular names and symbols or for base six. They do work well, however, and are relatively free from difficult or suggestive combinations of sounds. Perhaps an even closer parallel to our English number names could be made by putting in such a word as "pop" for the name following "lum." "Lalum," which is really analogous to "onety-one," is much easier than our eleven or twelve which do not follow the system very closely.

Half the fun and possibly more than half the learning is in exploring and inventing. Even our harassed college students enjoyed it. And no one who has invented a set of suitable words and symbols and who has worked examples in his own system will remain innocent of the working of ordinary arithmetic.

Tease them to think

THOMAS J. JENNINGS *Garden City, New York*

Mr. Jennings teaches sixth grade in Stratford Avenue School in Garden City, New York, and is chairman of the Village-Wide Mathematics Committee. During the summer months he is a demonstration teacher at Hofstra College in Hempstead, New York.

Individualizing instruction is a constant concern of all teachers. The following article offers no panacea, but it does suggest a method by which youngsters, in spare moments, may explore a fascinating area in arithmetic—an area that might otherwise be used for less profitable pursuits.

Most youngsters enjoy the challenge of riddles and puzzles. The following program takes advantage of this interest.

Seventy-five brain-teasers were assembled from a number of sources. Each of these was written with India ink on its own $8\frac{1}{2}'' \times 11''$ sheet of yellow construction paper. Next, the sheets were arranged in order of increasing difficulty. Several of the easier brain-teasers were interspersed among the more difficult ones to provide renewed interest as the class advanced.

Once arranged, the brain-teaser sheets were numbered consecutively, and prepared for display on a convenient class bulletin board. The title "Have You Tried These?" was posted above the brain-teasers, which were displayed four at a time.

Sheets of graph paper were mounted on construction paper and placed on the same bulletin board, to serve as a record of brain-teasers solved by each youngster in the class. The names of the boys and of the girls were listed on separate sheets, to capitalize on friendly rivalry. Across the top of the sheets were listed the brain-teaser numbers.

Youngsters interested in answering brain-teasers listed their names on a reserved space at the corner of the blackboard. In spare moments before school, or at other opportune times, each youngster who had signed his name was given the chance to give his answers in private. When an answer was incorrect, he was encouraged to try again. For those brain-teasers that he had solved, a check mark was entered after his name in the appropriate column.

When a youngster solved a brain-teaser, he was "sworn to secrecy." He was not to share the answer, nor was he to give any hint as to the correct solution. Such secrecy was vital to the success of the plan. The youngsters joined in readily.

Hearing and recording answers in the manner indicated could become a time-consuming activity for the teacher. To avoid this problem, and to involve the class as much as possible, a simple plan was used. The brain-teasers were divided into groups of ten. The youngster who was first to answer correctly all ten in one group became the "expert." He was then responsible for hearing and recording correct answers for that group of ten brain-teasers. You can imagine the glow of pride when a youngster was announced as the "expert" for a group of teasers. As each successive group of ten teasers was posted and solved, another youngster became an "expert."

Brain-teaser cards may be changed as often as seems desirable, though a maximum of twice a week is suggested if four are displayed at once. If the cards are changed too frequently, the slow-starters tend to be discouraged by the backlog of cards to be solved.

When cards were removed from the display board, they were stored in a folder so youngsters could have a chance to review any they had not solved previously. This folder was used most in the period before school sessions.

Care must be taken to avoid invidious comparisons between children on the basis of their success with the brain-teasers. Those who show little interest in the program can often be helped to overcome their shyness by a few private words of encouragement from the teacher. Once a hesitant youngster has solved a brain-teaser or two, he usually becomes one of the group that waits anxiously for new brain-teasers to be put on display.

This is the third year that the program has been used with a sixth-grade class. The values to the class are many:

1. The youngster must call upon intuition and ingenuity, because many of the brain-teasers call for skills not yet taught in class.
2. Youngsters with initiative and ability are challenged to use their abilities to the fullest.
3. Old learnings in arithmetic and mathematics are applied in new ways.
4. Youngsters learn to read material with greater care when they find that hasty reading has caused them to misinterpret the question asked.
5. Many youngsters are stimulated to search out additional brain-teasers to add to the collection.
6. Some youngsters take a real interest in arithmetic for the first time.
7. A community of spirit develops among those who have solved difficult brain-teasers that would not develop in the ordinary class program.

The sources of brain-teasers are numerous. Daily newspapers often carry questions or puzzles that are appropriate. Children's magazines and newspapers are good sources. *The Magic House of Numbers*, written by Irving Adler and published by Day, provides many fine teasers, as does, *Fun with Puzzles*, written by Joseph Leeming and published by Lippincott.

It takes only a little time to institute this enrichment program and the results are most gratifying.

Can you do it?

Nine Dots. Here is an old puzzle. The diagram shows nine dots in the form of a square. Draw four straight lines so as to cross out every dot. Do not cross any dot more than once, nor retrace any line, nor lift the pencil from the paper until all nine dots have been crossed.

—Taken from Geoffrey Mott-Smith, *Mathematical Puzzles for Beginners and Enthusiasts*, 2d rev. (New York: Dover Publications, Inc., 1954), p. 4.

Individualizing arithmetic-teaching

EUGENE R. KEFFER
Central Missouri State College, Warrensburg, Missouri
As a member of the Division of Education and Psychology, Dr. Keffer supervises the Off-Campus Student-Teaching Program at Central Missouri State College.

Evidence available from general aptitude, special aptitude, and achievement tests and observations by highly skilled teachers of arithmetic leads us to conclude that not all pupils have the ability to master arithmetic skills according to a set schedule. And yet, quite frequently all pupils in any given class are taught simultaneously the same arithmetic principles and processes.

Readiness to learn is determined largely by the sum of all the characteristics which make an individual more likely to respond positively in terms of the learning situation. It involves the pupil's mental equipment, his goals and needs, and his learned ideas and skills.[1]

While it is not our purpose to minimize the importance of a pupil's needs and goals as conditioners of readiness, in this article we are more concerned with aptitude and achievement as readiness factors in learning. Maturation, the natural growth of the *physical bases* for mental functions, conditions the *ability* to learn. Where there is insufficient maturation, then, it may be wholly impossible for a child to utilize the related mental functions. Mental development, the progressive growth and organization of the mental functions and psychological behavior of the individual, is conditioned by both maturation and learning. Consequently, former experiences, maturation of the somatic bases of learning, and previous exercise of functions determine, to a large extent, the degree of readiness which a pupil possesses.[2]

Experiments in which standardized psychometric devices have been used to gather the data have demonstrated to us that there are major differences among the pupils in the typical classroom of an American school. Among the differences which have been isolated are the following: mental age, level of achievement, and specific aptitudes.

In this paper, we propose a method of teaching arithmetic which takes cognizance of the aforementioned factors and which is organized in such a manner that each pupil is encouraged to develop arithmetic skills in accordance with his readiness and application. This method has been used very successfully for many years by Mrs. Bertha A. Hopkins, sixth-grade teacher in the College Laboratory School of Central Missouri State College.

In this teaching method, the classroom is organized as a laboratory. During the first week of school, the teacher attempts to ascertain the level of achievement of each pupil in the class. The criteria which are used to ascertain the level of achievement are: teacher observation of each individual pupil's work, teacher-made and textbook tests, and conferences with each pupil to determine the highest achievement level at which he understands the arithmetic principles and processes. Each pupil, then, is given his initial assignment

[1] Lee J. Cronbach, *Educational Psychology* (New York: Harcourt, Brace & Co., 1954), p. 74.

[2] Howard L. Kingsley, *The Nature and Conditions of Learning* (New York: Prentice-Hall, Inc., 1946), pp. 31, 56.

at the achievement level where he can work with understanding.

Before assignments are made pupils are urged to calculate problems which they can work independently. Textbooks at the fifth-, sixth-, and seventh-grade levels are made available to them. Enrichment materials are provided in order to keep all pupils busy during this period of pupil classification.

After all pupils are classified, instruction and assignments are given. Each individual at a specific level of achievement is invited to the desk of the teacher who explains and illustrates the principles and procedures necessary to solve problems at this level. Then the pupil works a problem on the blackboard to demonstrate his knowledge and understanding of principles and procedures involved. When the individual has demonstrated that he knows and understands these principles and procedures, he is given an assignment which does not go beyond the principles and procedures he has learned.

If two or more individuals are at the same level of attainment at any given time, the teacher instructs them, at her desk, in the same manner that she used with the individual pupil. Each individual in the group must demonstrate his knowledge and understanding of the principles and processes involved before any assignment is given to him.

During the regular class period, following the assignment, the learner works the problems at a speed which he chooses until he completes the assignment or until he doesn't understand what he is doing. He keeps a spiral notebook in which the problems are organized so that the teacher can spot-check them readily. From this spot-checking the teacher determines his progress, his understanding of the problems, and the areas where more understandings need to be developed.

When the teacher is not explaining a new process during the class period, she circulates among the pupils and checks their notebooks to determine the degree of mastery and understanding which they have attained relative to the problems they are working. If they need drill exercises in order to develop mastery and/or understanding of the problems, these exercises are assigned; otherwise, no drill exercises are assigned. If review or relearning is indicated, individual instruction is arranged. Thus, valuable time is conserved which is used to relearn processes and/or principles which were forgotten or to learn new principles and processes when mastery is achieved on a given assignment.

Occasionally the entire class may be taught the same principle(s) and/or process(es) simultaneously. The entire class is taught simultaneously, however, only if all the pupils have the backgrounds to understand the principle(s) and process(es) being taught. For example, the pupils may be constructing a map of Italy during their study of the geography of the land, and lack knowledge and understanding of Roman numerals, area, perimeter, and addition and substraction of measurements. In this instance the teacher would instruct the pupils simultaneously in these areas of deficiency.

The teacher must be very familiar with the arithmetic taught in grades four, five, six, seven, and eight in order to plan for the needs of each individual pupil. She may have to teach pupils the principles and processes which they have failed to learn or have forgotten, and she may find it necessary to teach principles and processes which are taught ordinarily in the seventh or eighth grades.

In order to provide materials for pupils, it is necessary to provide textbooks by several authors and publishers for each of the grade levels indicated by the levels of achievement represented in the class. Although one particular textbook is chosen for each grade level as the "textbook" for that grade level, problems in it are not reassigned if mastery is not attained when they are worked the first time. Rather, similar problems in another textbook are assigned for drill and/or relearning.

When instruction is individualized, it is very necessary that evaluation of pupil progress be continuous. In addition to continuous evaluation, each pupil is given a written test in problem-solving and computation at the end of each unit or chapter that he completes. The teacher checks the procedures and answers and returns the test to the pupil, who is instructed to check the problems and locate his errors if there are any. If he can't find them, the teacher identifies them for him. Then she assists the child to correct the errors. Where he lacks understanding, the work is repeated until mastery and understanding are attained. After the pupil masters the principles and processes, he is given individual instruction in a new process and/or principle and is given a new assignment.

In order to control feelings of inferiority or superiority, the teacher discusses with the pupils the various degrees and kinds of abilities which are expected in the classroom. She uses the physical growth pattern to explain physical variations, and explains that both physical and mental growth vary in degree and *in toto*, and that the amount of progress which a pupil makes is dependent, in part, upon these growth factors. These pupils are taught to accept their limitations and their strengths. They are taught to be dissatisfied if they don't try to make progress commensurate with their abilities, and not to feel inferior or superior if they are above or below the level of attainment of their classmates. No feelings of inferiority or superiority have been detected by the teacher. When pupils are very close in their assignments, some competition may develop. The competition seems to stimulate the persons involved to progress more rapidly than they would under other circumstances.

We have discussed a laboratory method of teaching arithmetic. With this method, each pupil works at the level of his understanding, and progresses at a rate commensurate with his readiness and application.

Are you planning to submit a manuscript?

(*Continued from page 241*)

For an article in a periodical:

PETERS, ANN C. "The Number System and the Teacher," THE ARITHMETIC TEACHER, IV (October, 1957), 155–60.

5. All drawings should be on good quality white paper in black India ink. Also, use India ink for letters and numbers which appear in the drawings. Drawing should be somewhat larger than the final reproduction in the periodical. When mailing, insert cardboard in envelope to prevent bending.

6. You will have an opportunity to order reprints when galley proofs are sent to you. Authors of feature articles will receive three copies of the journal from the editorial office.

7. Changes made in galley proofs that are sent to the author for proofreading are expensive. Indicate only those corrections which are in agreement with your manuscript.

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E. GLENADINE GIBB, EDITOR
THE ARITHMETIC TEACHER
Iowa State Teachers College
Cedar Falls, Iowa

A third grade enjoys arithmetic

Editor's note

Acknowledgment is made to Mrs. Mildred Savaides, third-grade teacher, Wilson School, Arlington, Virginia, and to her boys and girls for descriptions of their classroom work and for samples of materials.

Elementary teachers throughout the country today are taking mathematics courses and participating in national institutes and local workshops in an effort to improve their competence in teaching mathematics to elementary children. While taking a mathematics course offered in the Washington area, Mrs. Savaides attempted to apply some of the mathematics from her course to the mathematics her third graders were learning. She was rewarded by the enthusiasm and interest of her third graders, who considered it quite a challenge to understand the underlying mathematical concepts involved in the mathematics they were learning to perfect their skill in processes.

The following illustrations are typical of the types of experiences Mrs. Savaides has been providing for her children. Illustrations recorded are copied from children's papers as they worked at the chalkboard or as work was explained by children.

Expansion of numerals

The children were asked to expand three-place numerals in as many ways as they could. The majority of children in the group were able to expand or to show the meaning of the numbers in these ways:

$$\begin{array}{r} 100 \\ 100 \\ 100 \\ 100 \\ 10 \\ 10 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ +\ \ 1 \\ \hline 427 \end{array}$$

$$427 = 400+20+7$$
$$(4\times100)+(2\times10)+(7\times1)$$
$$(4\times10\times10)+(2\times10)+(7\times1)$$
$$(4\times10^{2})+(2\times10^{1})+(7\times10^{0}).$$

Adding and multiplying three-place numbers

The children had achieved an excellent understanding of regrouping in addition by expanding numbers to be added, adding, and then regrouping ten or more ones to tens and ones, or ten or more tens to hundreds and tens:

$$\begin{aligned} 324 &= 300+20+4 \\ 245 &= 200+40+5 \\ 426 &= 400+20+6 \\ &\quad\overline{} \\ &\quad 900+80+15 \\ &= 900+90+5 = 995. \end{aligned}$$

They also used a four-step procedure which involved adding and checking. In Step I they used a simple expansion of the addition process, recording the answers for ones, tens, and hundreds separately,

and then combining the three partial answers. Step II was the short cut, or "carrying" process. In Step III they switched addends and repeated Step I. In Step IV they used the short cut with switched addends.

Step I	*Step II*
Expand	Regroup
	11
324	324
+279	+279
13	603
90	
+500	
603	

Step III	*Step IV*
Switch Addends	Regroup
Expand	
279	279
+324	+324
13	603
90	
+500	
603	

The language the children used definitely aided their understanding. It was evident that they were aware of the place value of each number and of the different ways numbers could be regrouped. To quote one child's explanation: "Thirteen ones can be regrouped into 1 ten and 3 ones; 10 tens can be regrouped into 1 hundred and 0 tens." Some of the children wrote carry numbers in Step II, others omitted them. Another child gave an account of his thinking as he expanded the numbers as shown and found the answer for this example:

584 = 5 hundreds, 8 tens, 4 ones
−326 = −3 hundreds, 2 tens, 6 ones.

"I have to regroup tens and ones before I can subtract in one's place."

5 hundreds, 8 tens, 4 ones
= 5 hundreds, 7 tens, 14 ones
−3 hundreds, 2 tens, 6 ones
2 hundreds, 5 tens, 8 ones

Although the children had learned only the "fives" in multiplication, by means of their knowledge of expanding numbers, they found multiplying a three-place number to be a rather simple procedure. Here are some samples of their work:

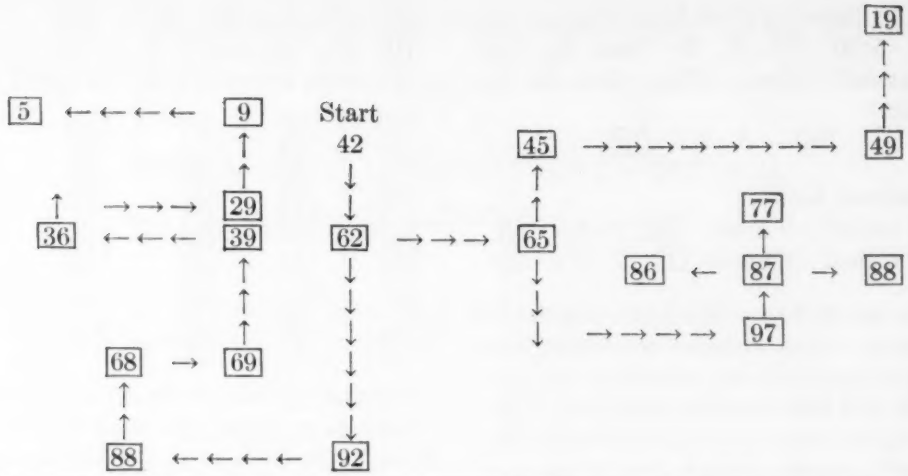
426	400+20+6
×5	×5
	2000+100+30
	100
	30
	2130

268	
×5	
40	40 ones or 4 tens
300	30 tens or 300
1000	10 hundreds or 1000
1340	

Arithmetic sentences

The children enjoyed making number sentences true by inserting the symbols =, > (is greater than), < (is smaller than), or ≠ (is not equal to). They were instructed to use > or < as needed, but they were warned that they could use ≠ only twice on the page. During oral checkup they indicated instances in which two signs were appropriate. Many children wrote answers under the sentences to check their choices:

$(3+6) > (2+5)$
9 7
$(7+4) = (4+7)$
11 11
$321 \neq 231$
$(3+4+6) < (4+4+6)$
13 14
$(3 \times 5) < (3 \times 6)$
15 18
$(3 \times 100) = (3 \times 10^2)$
$(6+4) < (32-20)$
10 12
$(7-7) \neq (7-1)$
0 6
$(36+30) < (46+30)$
66 76
$(640+240) > (640+204)$
880 844



V+V+V= ☐

7000
600
40= ☐
7
9,826< ☐

XXV= ☐+☐+☐

☐+☐ < 9+8

8+7> ☐+☐

3+7< ☐+☐

(2×5)+6= ☐

(3×5)+4= ☐

Arrow puzzle patterns

The children were asked to think about the hundred chart in their room, which was set up in the form shown below, and find the missing numbers:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

Arrow rules agreed upon by the children were:

1. ↑ = number becomes 10 smaller.
2. ↓ = number becomes 10 larger.
3. → = number 1 larger.
4. ← = number becomes 1 smaller.
5. When the end of a row is reached continue to think one larger or one smaller as if all the numbers were stretched out along a single line.

Above is the arrow puzzle the children solved by filling in the empty boxes.

Clock arithmetic

From their experiences with clock arithmetic, the children learned that when adding hours, 9+6 was not 15 but 3! Below are excerpts from a worksheet on clock arithmetic:

To *add hours*, go to *the right* around the clock. To *subtract hours*, go to *the left* around the clock.

1. If school starts at 9:00 o'clock, what time will it be in 4 more hours?

   1 9+4= 1

2. Add hours:
   11 hours+3 hours= 2
   8 hours+6 hours= 2
   8+2= 10
   11+5= 4

3. Bill's father arrived home from a trip at 5:00 o'clock. He said he had traveled 7 hours. What time did he start?

[10] 5−7= [10]

4. Subtract hours:

4 o'clock−6 hours [10] 6−8= [2]
12 o'clock−8 hours [4] 7−7= [2]

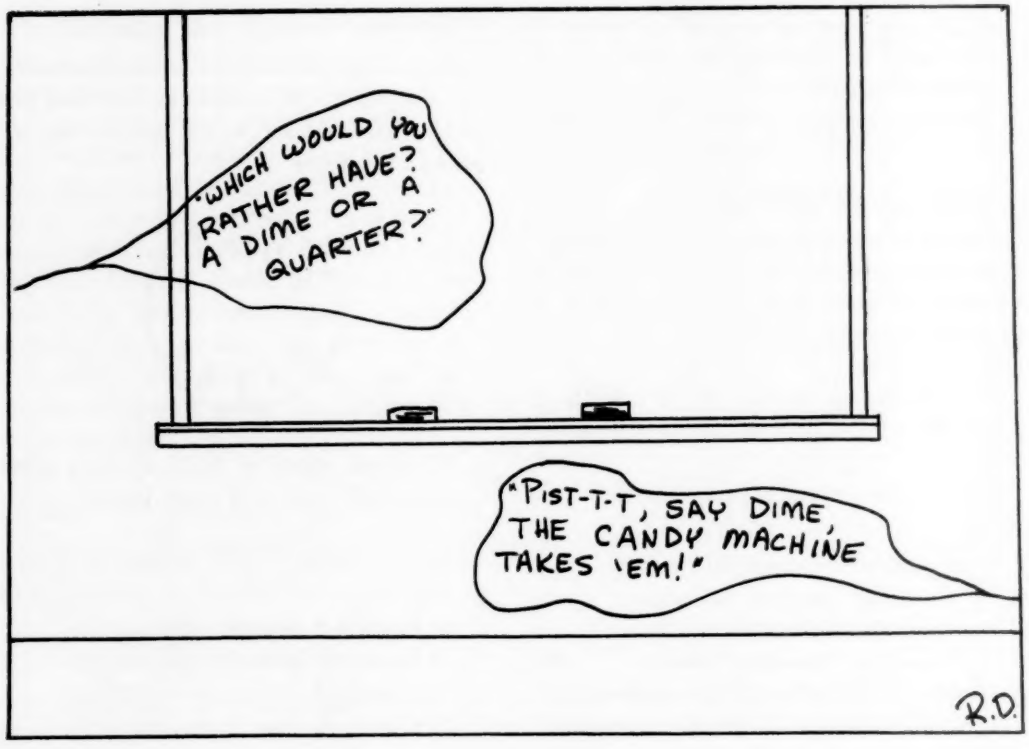
The puzzle below illustrates a practice technique which includes combining several processes and the immediate application of new learning. The preceding week, the children had been introduced to the "fives" in multiplication, which are now incorporated into their practice puzzle.

[A]$_5$ [B]$_7$ [C]$_{10}$ [D]$_7$ [E]$_8$ [F]$_7$ [G]$_{13}$ [H]$_0$ [I]$_{10}$

[A] $(5\times4)+(4)-(5\times3)+(4)=$ [5]
[B] $(5\times6)+(7)-(5\times5)+(5)=$ [7]
[C] $(3\times5)+(8)-(5\times2)+(3)=$ [10]
[D] $(3+4+7)-(2+2+3)=$ [7]
[E] $(4+7+3)-(1+3+2)=$ [8]
[F] $(5+4+5)-(14-7)=$ [7]
[G] $(18-9)+(24-20)=$ [13]
[H] $(36-24)-(24-12)=$ [0]
[I] $(90+10)-(40+10+30+10)=$ [10]

[A] + [B] + [C] = [22]
[D] + [E] + [F] = [22]
[G] + [H] + [I] = [23]
[A] + [D] + [G] = [25]
[B] + [E] + [H] = [15]
[C] + [F] + [I] = [27]
[A] + [E] + [I] = [23]
[C] + [E] + [G] = [31]

Certainly much of the mathematics included in a course for teachers does not have application in a third-grade classroom. It merely provides background for enriching the teacher's understanding. Some of it does apply, however, as is evident from the work of the children in one third-grade classroom. It seems reasonable to assume that mathematics for children will be enriched to the extent that applications which are within the grasp of children can be found and adapted to the appropriate level of instruction.



Research on elementary-school mathematics—1960

This report thus far represents approximately 40 per cent of the complete research. The last 60 per cent will appear in the October issue.

This summary of research on elementary-school mathematics represents a continuation of a series begun four years ago in THE ARITHMETIC TEACHER and brings this series up to date through the calendar year 1960.[1]

The present summary is devoted to *published* research reports; for *doctoral* theses or dissertations, we refer our readers to *Dissertation Abstracts* (as distinct from other published research reports).

The annotated references in this summary include accounts of research on elementary-school mathematics (or closely related thereto) that were reported in periodicals, monographs, and the like during the calendar year of 1960. A serious attempt was made to prepare a comprehensive listing rather than a selected one.

In preparing the 1960 summary, the writer applied the same criteria of delimitation that were used in previous years. Thus, published references included in the bibliography are restricted to (1) normative and experimental studies which report specific data or findings on a problem associated with, or closely related to, mathematics instruction in the elementary school; and (2) bibliographies, summaries, and more or less critical discussions which relate, in whole or significant part, to such normative and experimental studies. It is hoped that these criteria were applied with reasonably acceptable fidelity when deciding whether or not to include a particular reference among those listed below in this 1960 summary.

1. American Association for the Advancement of Science, Science Teaching Improvement Program. *Study on the Use of Special Teachers of Science and Mathematics in Grades 5 and 6.* Washington, D.C.: The Association, 1960, 15 pp.

Following a brief presentation of relevant background on the "one-teacher" and "special-teachers" plans of elementary-school organization, this report describes the purposes and general nature of a study now in progress in four co-operating school systems (Cedar Rapids, Iowa; Lansing, Michigan; Washington, D.C.; and Woodford County, Kentucky) on the use of special teachers of science and mathematics in Grades 5 and 6. In the future, look for other reports which give more detailed information on the design of the study, its findings, conclusions, and recommendations.

2. Barnes, Kenneth; Cruikshank, Raymond; and Foster, James. "Selected Educational and Experience Factors and Arithmetic Teaching," THE ARITHMETIC TEACHER, VII (December, 1960), 418–420, 430.

This article reports a study based on a sample of 102 of the 157 fourth-grade teachers in the ten school districts of San Mateo and Santa Clara (California) counties. In terms of the elementary-school principals' ratings of these teachers on a trichotomous "superior-good-fair" basis, it was found (among other things) that "No significant relationship was demon-

[1] For previous summaries in this series, covering the calendar years of 1951 through 1959, see THE ARITHMETIC TEACHER for April, 1957; April, 1958; April, 1959; and May, 1960.

strated between number and type of high school mathematics courses completed and the quality of the teachers' arithmetic instructional programs at the fourth-grade level." "No significant relationship was demonstrated between the number of college mathematics courses taken and the quality of the teachers' arithmetic instructional programs at the fourth-grade level." "The superior arithmetic teachers demonstrated an appreciably higher degree of interest in their high school mathematics courses than either the good or the fair teachers." "Years of teaching experience improve the quality of arithmetic instruction."

3. Bjonerud, Corwin E. "Arithmetic Concepts Possessed by the Pre-school Child," THE ARITHMETIC TEACHER, VII (November, 1960), 347–350.

Used as criterion measures, two specially constructed tests, one oral (fifty-seven items) and the other written (forty-five "picture" items), are presented by the author. Original study is based on a sample of one hundred children entering kindergarten in the fall of 1957; the study is repeated with sample of twenty-seven children entering kindergarten in February 1960. "The data secured from both groups of children . . . were strikingly similar. In generalizing the findings, both groups will be treated as one." Reports numerous findings, along with educational implications.

4. Brown, Kenneth E., and Kinsella, John J. *Analysis of Research in the Teaching of Mathematics: 1957 and 1958.* (Bulletin 1960, No. 8. U.S. Department of Health, Education, and Welfare; Office of Education.) Washington, D.C.: Government Printing Office, 1960, 50 pp.

A section (pp. 12–15) of this biannual research summary is devoted to "Research in the Teaching of Elementary School Mathematics," in which findings from twenty-two studies are presented in relation to thirteen key questions.

5. Brownell, William A. "Observations of Instruction in Lower-Grade Arithmetic in English and Scottish Schools," THE ARITHMETIC TEACHER, VII (April, 1960), 165–177.

This is a provocative report on "experimental" arithmetic programs (with particular attention to the Cuisinaire approach) in lower-grade English and Scottish schools, based on personal observation and on data from research studies. It concludes that, in relation to arithmetic instruction in the lower grades of American schools, "We have seriously underestimated the attention span of school beginners." "Likewise, we have seriously underrated the 'readiness' of school beginners for systematic work in arithmetic." "We can safely ask children in the lower grades to learn much more in arithmetic than we are now asking them to learn . . . *if we wish* to do so." Three related critical questions are raised and discussed: "How important is arithmetic in the elementary curriculum?" "Should we, or should we not, undertake to develop in young children the mature kinds of mathematical skill and relationship now taught in some of the British experimental schools?" "In order to achieve what we may desire by way of a changed arithmetic program, to what extent need we substitute radically new materials and methods for those in common use?"

6. Burns, Paul C. "Arithmetic Books for Elementary Schools," THE ARITHMETIC TEACHER, VII (March, 1960), 147–149.

This article summarizes findings from an analysis of ten arithmetic textbook series and their accompanying workbooks and teacher manuals or guides.

7. Burns, Paul C. "Intensive Review As a Procedure in Teaching Arithmetic," *The Elementary School Journal*, LX (January 1960), 205–211.

Here a study is made " . . . to determine whether the use of thought-provoking review questions, spaced advantageously throughout instruction in sixth grade, would result in greater achievement in common and decimal fractions."

Data were derived from 337 pupils, divided between seven control and seven experimental classes. Control-group teachers followed their usual review procedures; experimental-group teachers used a specially prepared series of ten sets of review study questions (twenty-four questions per set). Two non-standardized tests (a fifty-item test on common fractions and a fifty-item test on decimal fractions) were used as pretests, as immediate-recall tests, and as delayed-recall tests. These data were supplemented by scores on the thirty items dealing with common and decimal fractions on a standardized test used as a pre- and end-measure. It was found that " . . . the pupils who had practiced with review study questions tended to handle the test exercises more effectively than the pupils who had had the same amount of instruction but had not worked on the review study questions. The difference in over-all mean gain or change was statistically significant [in favor of the experimental classes]."

8. Buswell, Guy T. "Arithmetic," in *Encyclopedia of Educational Research*. (3rd ed.). New York: Macmillan Co., 1960, pp. 63–77.

This article summarizes general findings from 140 research studies and reports on elementary-school mathematics.

9. Chase, Clinton L. "The Position of Certain Variables in the Prediction of Problem-Solving in Arithmetic," *Journal of Educational Research*, LIV (September, 1960), 9–14.

This work makes a study of factors related primarily and secondarily to the arithmetic problem-solving ability of 119 sixth-grade children (58 girls, 61 boys). The problem-solving section of the Iowa Every-Pupil Test of Arithmetic is used as the criterion measure, with fifteen independent variables studied in relation thereto. It identifies only three of the fifteen independent variables as primarily related to problem-solving ability: arithmetic computation, reading to note details, and fundamental knowledge of arithmetic. These three factors accounted for the following respective proportions of the variance in problem-solving scores: 32 per cent, 16 per cent, and 14 per cent, with 38 per cent of the variance unaccounted for by these three primary variables. Further analysis identified the variables associated with each primary factor in a secondary relation to problem-solving ability.

10. Corle, Clyde G. "A Study of the Quantitative Values of Fifth and Sixth Grade Pupils," THE ARITHMETIC TEACHER, VII (November, 1960), 333–340.

This study seeks to determine how accurately 108 sixth-grade pupils (55 boys, 53 girls) and 39 fifth-grade pupils (17 boys, 22 girls) could estimate and then measure each of 10 quantities; and to determine the relation of these abilities to over-all arithmetic achievement. Findings are reported in terms of several different indices or measures of estimation and measurement ability, with differences analyzed by grade level and by sex. The article points to the need for more specific and explicit attention to the development of estimation and measurement abilities in connection with systematic programs of elementary-school mathematics instruction.

11. Dooley, Mother M. Constance. "The Relationship Between Arithmetic Research and the Content of Arithmetic Textbooks (1900–1957)," THE ARITHMETIC TEACHER, VII (April, 1960), 178–183, 188.

Based on a study of relevant research since 1900 and on a study of 153 arithmetic textbook series published in the United States since 1900, this work seeks to determine the effect of research upon content and method as reflected in children's textbooks. For the twelve topics investigated, nine were found for which the research effect was "direct and immediate" and three in which research recommendations were rejected somewhat. The conclusion is that "When the [research] recommendations were clear, concise, and exact, they were incorporated into many

textbooks within five years." "When the recommendations were general, intangible, or based upon subjective data, they were not applied as rapidly as where the findings were precise and well-supported by adequate data. Lack of clearness and explicitness in the presentation of recommendations led to slowly developing trends." "Those recommendations which were published in the Yearbooks of the National Society for the Study of Education and the National Council of Teachers of Mathematics tended to be applied very quickly."

12. Erhart, M. "Arithmetic for the Academically Talented," THE ARITHMETIC TEACHER, VII (February, 1960), 53–60.

Mr. Erhart summarizes findings from a 17-page questionnaire returned from 115 of the 500 Educational Records Bureau member schools to whom it was sent in an attempt to answer two broad questions: (1) "What administrative devices seem most appropriate for the instruction of the academically-talented in arithmetic?" (2) "What curriculum, teaching materials, and methods are being used in the arithmetic programs in E.R.B. schools, and what are the opinions of the instructors as to their effectiveness?" The sample of 115 schools studied embraces 954 classes in Grades 1–8, in which the median class size per grade ranges between 15 and 18 pupils and the median Kuhlman-Anderson I.Q. per grade ranges between 120 and 128.

13. Flournoy, Frances. "Meeting Individual Differences in Arithmetic," THE ARITHMETIC TEACHER, VII (February, 1960), 80–86.

This article reports on an exploratory investigation in which a group of thirty-six teachers assessed the advantages and disadvantages of using certain teaching variations in different types of class organization. Variations in learning time, in content studied, and in teaching methods and materials were explored in class situations ranging from those using class-as-a-whole procedures, at one extreme, to those using completely individualized instruction, at the other extreme. The report also includes an analysis of procedures used in six arithmetic textbook series for aiding teachers in meeting individual differences.

14. Folsom, Mary. "Teachers Look at Arithmetic Manuals," THE ARITHMETIC TEACHER, VII (January, 1960), 13–18.

This work gives portions of a study of teachers' uses of and reactions to the manuals or guides accompanying arithmetic textbook series, based on direct observation of actual classroom practice coupled with information from a questionnaire. It concludes with an extended, provocative discussion of the study's implications for the preparation of manuals or guides in the future.

15. Fulkerson, Elbert. "How Well Do 158 Prospective Elementary Teachers Know Arithmetic?" THE ARITHMETIC TEACHER, VII (March, 1960), 141–146, 158.

Here data is given from a forty-item test constructed by the investigator and administered to students enrolled in a college course devoted to a "professional treatment of the subject matter of arithmetic methods and a study of trends and current literature on the teaching of arithmetic." Item-analysis information is given, along with analyses of performance on the test in relation to teaching experience, classification in college, and preparation in high school and college mathematics. It was found (among other things) that "Students with teaching experience did significantly better than those without." "Performance becomes progressively better as the level of classification in college becomes higher." "Performance becomes better as preparation in mathematics increases." Concluded that " . . . far too many of the 158 prospective elementary teachers studied herein have an insufficient knowledge of arithmetic to teach the subject effectively."

16. Gunderson, Richard O., and Feldt, Leonard S. "The Relationship of Differences Between Verbal and Nonverbal In-

telligence Scores to Achievement," *Journal of Educational Psychology*, 51 (March, 1960), 115–121.

This is a study based on a sample of one hundred fourth-grade pupils divided into twenty-five quartets matched on total I.Q. (California Short-Form Test of Mental Maturity) but differing in the relation between language I.Q. and nonlanguage I.Q. They differ in amounts ranging from twenty-four or more points in favor of the language I.Q. to an equal disparity in favor of the nonlanguage I.Q. Achievement of all pupils was assessed by a standardized test (Iowa Tests of Basic Skills), supplemented by a thirteen-item questionnaire administered to each child's teacher. The four groups were found to differ significantly in arithmetic achievement (as well as in reading, oral and written language, vocabulary, and work study skills), with highest achievement in the group with marked verbal-I.Q. superiority and lowest achievement in the group with marked nonverbal-I.Q. superiority. However, the differences in arithmetic achievement among the groups were fewer than were the achievement differences in reading, language, and vocabulary.

17. Hartung, Maurice L. "Selected References on Elementary-School Instruction: Arithmetic," *The Elementary School Journal*, LXI (December, 1960), 167–170.

This familiar annual annotated listing includes twenty-four references.

18. Hudgins, Bryce G. "Effects of Group Experience on Individual Problem Solving," *Journal of Educational Psychology*, LI (January, 1960), 37–42.

As an extension of research studies which have found that children working together in groups are more successful in solving problems than are comparable children working as individuals, this study was designed to test three hypotheses: (1) " . . . that problem solving experience in a group improves individual ability more than does individual experience." (2) " . . . that individual ability to solve arithmetic problems improves as a result of specifying the steps involved in arriving at solution." (3) " . . . that the improvement of problem solving ability as a result of group experience stems from the relevancy of intragroup communication to the processes involved in problem solution. That is, students tested individually following a period of time in which they had work as members of a group using specification would solve significantly more problems than subjects who worked initially in nonspecification groups." Data were derived from 128 fifth-grade pupils, divided into four groups equated on general mental ability (California Short-Form Test of Mental Maturity) and arithmetic achievement (California Arithmetic Test). (The experimental design will not be summarized in this present annotation.) It was found that " . . . although groups of students working co-operatively solve more problems than comparable students working alone, there is no significant improvement in the problem solving performance of the former *Ss* [i.e., pupils working in groups] because of this group experience. Consequently, the first hypothesis ["that problem solving experience in a group improves individual ability more than does individual experience"] was rejected. Nor were the other two hypotheses supported; that is, practice in specifying the steps used in solving a problem did not improve performance any more than in the case where such specification was not made. This was true whether specification was used by individuals or by small groups." In connection with his discussion of the study's findings, the investigator conjectures that "The most reasonable hypothesis at this point appears to be that the problem solving superiority of small groups depends upon the efforts of the most able member of the group to communicate his knowledge to others, and upon the degree to which he achieves acceptance of his solutions. If this hypothesis proves tenable, transfer from the group to the individual situation would not be expected to occur."

19. Jones, Emily. "Historical Conflict —Decimal Versus Vulgar Fractions," THE ARITHMETIC TEACHER, VII (April, 1960), 184–188.

This article presents a condensation of findings from the investigator's historical-documentary study based on an analysis of 110 arithmetic textbooks published in English in America from 1719 to 1839.

20. Josephina, Sister. "Mental Arithmetic in Today's Classroom," THE ARITHMETIC TEACHER, VII (April, 1960), 199–200, 207.

This work reports a small-scale experiment designed " . . . to study two techniques of presenting mental arithmetic problems. Did the pupils perform better when they read the problem from the workbook and looked at it while mentally solving it, or would more achievement result if the pupils listened to the problem read to them by the teacher?" Data were derived from two groups of twenty-nine fifth-grade pupils, equated on I.Q. One group followed the "teacher-reading" method for four weeks, then the "pupil-reading" method for four weeks. During the same period of time, the other group used these two procedures in reverse order. Significantly greater achievement was found for the pupil-reading method.

21. Keough, John J. "The Relationship of Socio-Economic Factors and Achievement in Arithmetic," THE ARITHMETIC TEACHER, VII (May, 1960), 231–237.

A study of the arithmetic achievement of 208 eighth-grade pupils in relation to the following factors is reported here: (1) trichotomous grouping of children ("enriched," "average," and "slow") by the guidance counselor for citizenship education and English instruction; (2) occupation of father; (3) birthplace of parents; and (4) newspapers read by the family.

22. Miller, G. H. "Time Spent on Arithmetic in Foreign Countries and in the United States," THE ARITHMETIC TEACHER, VII (May, 1960), 217–221.

This article presents data on the amount of time (minutes per day) allotted to arithmetic instruction in each of the elementary grades (1–6) in a composite of thirty-two foreign countries. For comparative purposes, it re-presents similar data from a previously published report by the investigator, which covers time allotments for arithmetic instruction in large city schools and in small schools in the United States. The following major conclusion is reached: "The other nations of the world devote more time to arithmetic than we do in the U.S.A." Based on the finding that "Over 85 per cent of the nations in this study required the same length of time in the first six grades," the investigator recommends that "There should be a specified time allotment for the instruction of arithmetic agreed upon by the U.S. school systems to provide a more uniform background for our pupils."

23. Phillips, Clarence. "The Relationship Between Arithmetic Achievement and Vocabulary Knowledge of Elementary Mathematics," THE ARITHMETIC TEACHER, VII (May, 1960), 240–242.

Data for the reported portion of a larger study were derived from administering the following tests to a sample of fifty-two college students enrolled in the course, "Arithmetic for Teachers": a non-standardized forty-item achievement test in elementary arithmetic, a fifteen-item vocabulary test of terms from elementary mathematics, and the California Short-Form Test of Mental Maturity. The following correlation coefficients were reported: 0.53 between achievement and vocabulary; 0.42 between achievement and mental maturity; 0.25 between vocabulary and mental maturity; 0.49 between achievement and vocabulary with mental maturity held constant; 0.39 between achievement and vocabulary for students above the median in mental maturity; and 0.61 between achievement and vocabulary for students below the median in mental maturity. (The difference between these last two coefficients was found to be not significant at the 5 per cent level.)

Books and materials

Exploring Mathematics on Your Own, Series of seven pamphlets with five in preparation, Donovan A. Johnson and William H. Glenn. St. Louis 26: Webster Publishing Company, 1960. Set of seven, $4.00.

This reviewer feels that the elementary teacher should regard these pamphlets as enrichment materials for those students whose interests and abilities reach beyond the typical topics and activities of the classroom. As such, they should prove effective in helping a student who is "exploring mathematics on his own." In contrast, the use of these materials by high school students for extra credit is stressed in a review of five of the pamphlets in the February, 1961, *The Mathematics Teacher*. The authors suggest that some of the pamphlets be used by an entire class as a textbook would be used.

Generally, each booklet is separated into parts which might be considered to be unnumbered chapters, consisting of several sections of exposition of concepts and sets of exercises. The expositions are generally not beyond the level of reading and vocabulary difficulty appropriate for those students in grades six through eight who will be motivated to study them. Some exercises are easy, and others will be considered difficult. On the whole, the students should find them interesting and challenging. The answer sections for all exercises can be removed, if desired. Each booklet, with one exception, contains a bibliography of additional readings on the subject; some of the references, however, are beyond the ability of even the best upper-elementary student to use profitably.

Sets, Sentences, and Operations may be a possible exception to the suggestion above, that these pamphlets be used only as enrichment materials. Since the development of the symbolism, concepts, and applications of sets to arithmetic, algebra, and geometry in this booklet is so similar to the development in units on sets, now included in some experimental materials for elementary schools, parts of this booklet, at least, can be used as text material. Students should find the selection of applications of set theory included here a mere whetting of the appetite for more of the same. The symbolism for unions and intersections is standard, and the use of Venn diagrams is exploited.

The Pythagorean Theorem includes accounts of what little is known historically about Pythagoras and his brotherhood, some appropriate consideration of number theory, various formulas for constructing Pythagorean triples, several of the more simple proofs of the theorem, various methods for extracting the square root of numbers, and a good variety of applications of the theorem, including a test of whether or not a triangle is a right triangle. Loomis' collection of over three hundred proofs of the theorem is mentioned, but otherwise three is no bibliography in this booklet.

The subject matter in *Topology—The Rubber-Sheet Geometry* stresses the idea that topological transformations preserve properties other than distance and size. The Königsberg bridge problem serves as a springboard into the exploration of networks and the subsequent development of Euler's formula relating numbers of vertices, arcs, and regions in the plane.

This is extended into three-dimensional space for figures such as the regular polyhedra. Many of the exercises are presented in tabular form for easy generalization. The usual topological tricks are also introduced.

Understanding Numeration Systems seems to be the booklet most appropriate for upper-elementary grades. After brief references to older systems such as the Egyptian, the expressions *digit*, *place value*, and *exponent* are established in reference to the Hindu-Arabic system, and then similar systems in bases five, twelve, and two are developed. For each base, notation, names, operations, and applications (especially for base two) are described and suitable exercises provided. That verbal subscripts, instead of numerical ones, are used to indicate bases is to be commended. The use of an abacus (simple construction directions provided), a nomograph, a number line, a slide rule, and a form of Napier's bones is described for operations in one strange base but easily extended to others. The booklet reaches its climax in a set of exercises where a strange set of digits is introduced for operations in base four using these digits, and the student is asked to provide a new set of number words as well.

Fun with Mathematics is a collection of puzzles, games, and tricks which have mathematical explanations, usually presented in this booklet in algebraic form. Some of the puzzles are of the form which requires one to think of a number, perform a series of operations, and then from the result the asker can deduce the original number. Other features of the booklet are a discussion of some calendar problems, such as finding the day of the week on which one was born, tricks using a clock face, and some tricks with cards and dominoes.

The first section of *Number Patterns* presents many examples similar to those found in most collections of mathematical oddities, where operations on numbers with certain patterns produce numbers with other patterns. Of more interest to the reviewer was the development that sums of sequences of odd integers, starting with one, produce the perfect squares. Similarly, the series and continued fractions, which have the value π as limits themselves, have patterns which, it seems, would fascinate many a student, even those in elementary school. Other topics touched upon include compound interest and magic squares.

The discussion in *Invitation to Mathematics* attempts to describe what it is that mathematics is about, how mathematicians go about their work, what the contribution of mathematics is and has been to our civilization and what frontiers mathematics has. It seems to the reviewer that this booklet meets this challenge as well as possible within the space of its relatively few pages. Some elementary-school students will no doubt feel less stimulation from this booklet than from some of the others; yet, many ideas developed in one of the other pamphlets are met again. For example, the appeal is again for pattern in the development of the numbers for Pascal's triangle. Similarly, the square pattern of some numbers met previously in connection with the Pythagorean theorem is here extended to rectangular arrangements for possible factor pairs; and then primes are defined as integers without such possibility of arrangement. New topics include modular arithmetic, inductive and deductive reasoning, and probability. The inculsion of biographical sketches of a few famous mathematicians is a stimulating and useful feature.

Elementary teachers and students should find other pamphlets in preparation by these authors equally stimulating and helpful. These include one each on graphing, measurement, and short cuts in computation.

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Seeing Through Arithmetic Tests for Grades 3–6, Maurice L. Hartung, Henry Van Engen, E. Glenadine Gibb, and Lois Knowles. Chicago: Scott, Foresman and Co., 1960. Specimen set, $1.00.

Achievement tests in arithmetic have been concerned primarily with testing the ability of a child to do isolated computations and to apply simple arithmetical skills to word problems. The curricular validity of these tests has been adequate, since instructional programs have stressed computation and this type of problem-solving. With the development in recent years of new materials and methods of instruction in arithmetic, the previously used tests are seriously lacking in validity and are rapidly becoming obsolete.

One of the series of tests designed to be valid for the modern approach to arithmetic is the Scott, Foresman *Seeing Through Arithmetic Tests*. A separate test has been developed for the third, fourth, fifth, and sixth grades, and is designed to measure achievement in each of six areas: (1) problem-solving, *selecting answers;* (2) *computation;* (3) problem-solving, *selecting equations;* (4) problem-solving, *solving equations;* (5) *arithmetic information;* and (6) *basic concepts*. These areas are carefully treated in the Scott, Foresman textbooks, and one who has used the texts should feel very much "at home" with the tests. On the other hand, children who have not used the texts would find the tests strangely different, particularly in Part 3. In this section, the objective is to select from several equations the specific equation which best represents the initial step in the solution of a problem. To draw this fine line between the initial step and subsequent steps would appear to present difficulties for children who have not been oriented to the Scott, Foresman approach.

Curricular validity for the Scott, Foresman materials has been built into the tests. Some statistical validity has been determined; however, additional analysis of some test items would be helpful. For instance, in Grade 3, Part 6, Item *D*, the concepts of object value, size value, and place value seriously overlap. The obvious answer is 12, although 142 is the more likely response. Since both of these answers fall into the category of "none of these," the item fails to discriminate. Item *N* of the same part resorts more to trickery than to a straightforward approach to the basic concept of place value. Presumably, the reliability of the tests is not mentioned in the guide, due to the fact that their newness has not permitted either retesting or the development of parallel forms. A report of reliability of the tests would appear to be desirable.

A consideration of the characteristic of usability serves to point up several advantages. The directions are simple, straightforward, and reduced to a minimum. Both the teacher's guide and the tests demonstrate the desirable characteristic of simplicity. Identical sample examples are repeated for each of the grade levels, resulting in a minimum of variation and brevity of instructions. However, it would seem that the re-use of samples in successive years might tend to induce a casual attitude on the part of children. Scoring is easy and is facilitated by a single template for each grade level. Shaded portions of the template are used effectively to draw attention to the appropriate section. The Cumulative Individual Record for a four-year period has the advantage of presenting a long-range view of pupil progress.

The format of the tests, though simple in design, tends to be monotonous. Obviously, the tests are not intended to be diagnostic, since very little scratch space is provided and children have the option of using separate scratch paper. The word "left" is used superfluously in several items of all four tests. Such multi-meaning words should be used with greater care. The suggested time limits for administering the tests are questionable. Forty-five minutes of formal testing on the second

part is in excess of the normal attention span of most third- and fourth-grade children.

The Scott, Foresman *Seeing Through Arithmetic Tests* are obviously the result of careful development and have both a high degree of curricular validity and a great deal of practical usefulness. The tests are excellent for the purpose for which they were intended; namely, to be used in connection with the Scott, Foresman texts as a measure of achievement in arithmetic.

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Books received

Arithmetic for the Modern Age, Aaron Bakst. New York: D. Van Nostrand Co., Inc., 1960. Cloth, 341 pp., $4.95.

Figures are Fun, series of five pamphlets with teacher's manual, J. A. H. Hunter. Toronto: The Copp Clark Publishing Co., Limited. Set of six, paper, $2.10.

Geometry, Part One: *Discovery by Drawing and Measurement*, W. H. E. Bentley and E. W. Maynard Potts. Boston: Ginn and Co., 1958. Cloth, 112 pp., $1.32.

New Ways in Numbers, Pupils' Books 1, 2 and 3, Right Rev. Msgr. Clarence E. Elwell, Sister Mary Stanislas, and Sister Mary Verone. Boston: D. C. Heath and Co., 1960. Paper, 254 pp., $1.36 each.

Understanding Arithmetic, Maytscherl W. Reckless. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1961. Cloth, 372 pp., $3.76.

In memoriam

Reeve, William David (1884–1961)

Dr. Reeve was editor of THE MATHEMATICS TEACHER from 1928 to 1949 and editor of the first twenty yearbooks published by the National Council of Teachers of Mathematics. Two of these yearbooks, the Tenth and the Sixteenth, were devoted to arithmetic.

His influence on the Council, on its publications, and upon the teaching of mathematics in the public schools of America is beyond the ability of the present generation to estimate.

Gugle, Elizabeth Marie (1878–1961)

On April 3, 1961, Miss E. Marie Gugle, a charter member of the National Council of Teachers of Mathematics and its fourth president, died at her home in Columbus, Ohio, at the age of 83. During her long professional career, she made a great contribution to mathematics education.

Every member of the National Council of Teachers of Mathematics, past or present, has profited from the wisdom and judgment of this distinguished lady.

State representatives, 1960-61

For many years our state representatives as a group have rendered invaluable service to our program. They have used a variety of methods, including personal solicitation, announcements at meetings, and sending reminders of "expirations" to promote membership in the council. They have advertised our publications, even to the extent in some cases of operating actual sales programs at meetings. They have announced council meetings and activities. In many cases they have been able to gather useful information and news from the field. Below is the list of representatives who are serving during the 1960–61 school year:

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Professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE ARITHMETIC TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

Joint Meeting with NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Twenty-first Summer Meeting

August 21–23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, C.S.B., St. Michael's College School, 1515 Bathurst Street, Toronto 10, Canada

Fortieth Annual Meeting

April 15–18, 1962
Jack Tar Hotel, San Francisco, California
Kenneth C. Skeen, 3355 Cowell Road, Concord, California

Other professional dates

Men's Mathematics Club of Chicago and Metropolitan Area

May 19, 1961
YMCA Hotel, 826 So. Wabash Ave., Chicago
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Twenty-First Summer Meeting

University of Toronto
August 20, 21, 22, 23, 1961

The twenty-first Summer Meeting of the National Council of Teachers of Mathematics will be held under the sponsorship of the Ontario Association of Teachers of Mathematics and Physics, at the University of Toronto, Toronto, Ontario, Canada, August 20–23, 1961. This is the first time that any meeting of the Council has been held in Canada.

Program high lights

GENERAL SESSIONS

Banquet

"Education in the 1960's"
Honorable John Robarts, Ontario Minister of Education

Luncheon

"Horizons Unlimited"
Roy O. Mertes, Director
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Others

"Principles of Learning and the Learning of Mathematical Principles"
Delox Wickens, Professor of Psychology, Ohio State University

"Advanced Mathematics from an Elementary Standpoint."
Ralph D. James, Professor of Mathematics, University of British Columbia

"On the Nature of Applied Mathematics"
Henry O. Pollak, Bell Telephone Laboratories, Inc.

Lectures

Nine lectures are scheduled for the purpose of improving or refreshing the mathematics background of teachers. Some of these are for secondary and college teachers and some are for junior high school and elementary teachers.

A two-session short course on the topic "Different Approaches to Exponential and Trigonometric Functions" will be given by Donald E. Richmond of Williams College, and a three-session short course on "Solution of Simultaneous Equations by Use of Matrices" will be given by E. F. Beckenback of the University of California at Los Angeles. Richard D. Anderson, Louisiana State University, will lecture on a topic related to geometry.

The junior high school and elementary teachers may hear a series of two lectures by Stanley Jackson, University of Maryland, on elementary geometry and by Burton H. Colvin, Boeing Scientific Research Laboratories, on a phase of arithmetic.

Section meetings

A series of programs is being sponsored by the Ontario Association of Teachers of Mathematics and Physics.

Speakers will describe the work of the Ontario Mathematics Commission and their experimental programs. Mr. Leonard Casiats and Mr. Hugh Fullerton of KCS Limited will discuss electronic computers and their use in business. Affiliated groups from New England, New York, Ohio, and Pennsylvania are sponsoring sectional meetings.

There will be a session concerned with "Teaching Machines." Professor E. G. Begle, Director, School Mathematics Study Group, and Professor Richard Alpert, Princeton, will report on "An Evaluation of Junior and Senior High School SMSG Materials" and "Attitudes toward Mathematics."

Also, there are many points of historical and aesthetic interest in Toronto, the second largest city in Canada. Visitors will enjoy such attractions as Old Fort York, the fabulous Casa Loma, the seaway facilities, beautiful parks, cathedrals, schools, public buildings, and fine residential districts. Tours of Toronto are planned for Sunday, August 20th.

Situated in Toronto is the renowned Canadian National Exhibition, which is the world's largest annual exposition. Since it will be in full swing at the time of the meeting, those attending the meeting will have the opportunity to join the three million people who visit it in its two weeks' duration.

Ontario is Canada's Variety Vacationland. Some may wish to stay to spend their vacations at one of the wonderful lake summer resorts.

Two post-convention tours have been planned: a one-day tour to Niagara Falls through the "Garden of Ontario"; and a two-day trip to Northern Ontario with a visit to the mining cities of Sudbury and Copper Cliff.

Program committee

Clarence Ethel Hardgrove, Clifford Bell, Mildred Cole, Eunice Lewis

Local chairman

Father John C. Egsgard, C.S.B.
St. Michael's College School, Toronto

Joint meeting with the NEA

With the co-operation of the Association of Mathematics Teachers of New Jersey, the NCTM will hold a meeting during the NEA Convention in Atlantic City, New Jersey, June 28, 1961. The meeting will open with a luncheon at noon at which John J. Kinsella, professor and chairman of mathematics education, New York University, will speak on the topic "What Is the Deadwood in the Mathematics Curriculum?"

At the General Session to follow at 2:00 p.m. there will be a panel discussion on "Experiences in Teaching Contemporary Mathematics." The panel will be composed of Florence Elder, chairman of the Mathematics Department, Junior-Senior High School, West Hempstead, New York and Gail Koplin, chairman of the Mathematics Department, Watchung Hills Regional High School, Plainfield, New Jersey. Following the panel discussion, Sheldon S. Myers, head of the Mathematics Section of the Test Development Division, Educational Testing Service, Princeton, New Jersey, will speak on the topic "Testing Programs and Contemporary Mathematics."

Harold Gouss and Bruce E. Meserve, president and president-elect respectively of the Association of Mathematics Teachers of New Jersey, will serve as presiding officers at the luncheon and General Session. The program has been arranged by a committee composed of Florence W. Borgeson, Evan M. Maletsky, Mary C. Rogers, and John A. Schumaker, chairman.

Those who wish to attend the luncheon should send reservations, with payment of $3.00 for each person, before June 10 to Mrs. Florence W. Borgeson, 307 Prospect Street, Westfield, New Jersey. The meeting will be held in the Shelburne Hotel.

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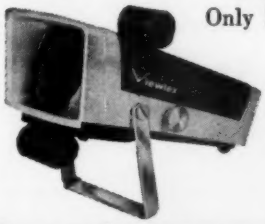

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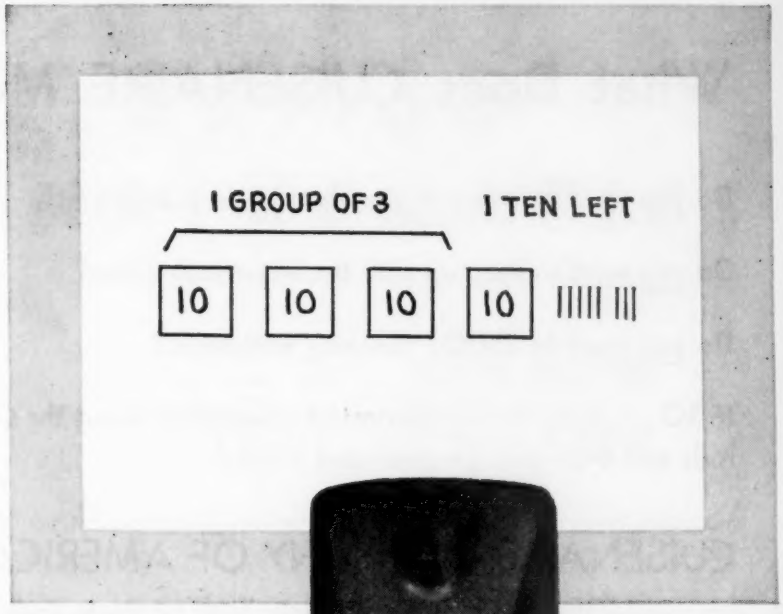

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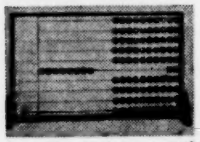

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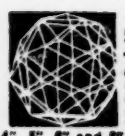

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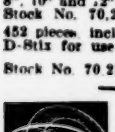

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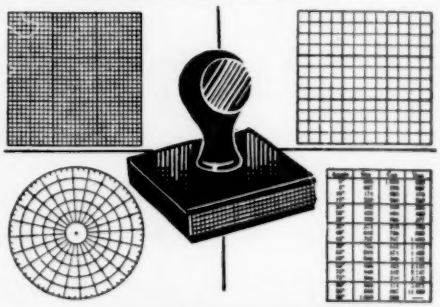

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